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Analytical design of a proportional-integral controller for constrained optimal regulatory control of inventory loop

Joonho Shin [a], Jongku Lee [a], Seungyoung Park [a], Kee-Kahb Koo [b], Moonyong Lee [c,*]

[a] Corporate R&D, LG Chem, Moonji-dong, Yuseong-gu, Taejon 305-701, Republic of Korea
[b] Department of Chemical and Biomolecular Engineering, Sogang University, Seoul 121-742, Republic of Korea
[c] School of Chemical Engineering and Technology, Yeungnam University, Kyongsan, Kyongbuk 214-1, Republic of Korea

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ABSTRACT

An analytical design method for a simple proportional-integral (PI) controller is developed for the optimal control of a constrained inventory loop. The proposed method explicitly deals with the important constraints in the inventory loop, such as the maximum allowable rate of change in the manipulated variable, the maximum allowable decay ratio and damping coefficient in the output response, as well as minimizing the optimal control specification. The simple and explicit form of the resulting tuning rule is clearly advantageous to practitioners.

1. Introduction

Inventory control loops are commonly encountered in the process industry. Typical examples include accumulator and bottom level control in a distillation column and the inventory control of a tank (Yang, Seborg, & Mellichamp, 1994). Inventory control is extremely important for the successful operation of most chemical plants, because it is through the proper control of the flows and levels that the desired production rates and inventories are achieved (Marlin, 1995).

Many studies have been conducted in an attempt to enhance the control performance of the inventory loop. Cheung and Luyben (1979) studied the liquid level control system with P-only and PI feedback controllers. They proposed a procedure with a design chart for the tuning of a PI controller in response to a step change of the inlet flow rate. However, it is quite complicated to determine the tuning parameters of a PI controller. Proportional-lag control (Luyben & Buckley, 1977) is a potentially good solution for liquid level control systems with feedforward compensation, but such feedforward control schemes (Luyben & Buckley, 1977; Wu, Yu, & Cheung, 2001) require an additional measurement which may be unavailable. Rivera, Morari, and Skogestad (1986) proposed the P-only controller using the internal model control (IMC) principle for the critically damped closed-loop response of a liquid level control system. Buckley (1983) discussed several nonlinear PI controllers to provide fast control action for large errors and slow action for small errors in the liquid level loops. MacDonald, McAvoy, and Tits (1986) proposed an interesting method of deriving an averaging level control algorithm to minimize the maximum rate of change of the manipulated flow.

In practice, the operation of an inventory control system should be located somewhere between the two extreme situations: the first, referred to as tight inventory control, is where the level is very important but any variation in the manipulated flow is not of great importance; the second, referred to as averaging inventory control, occurs when some variation in the level is acceptable as long as the value remains within specified limits, but the manipulated flow should not experience rapid variations of a significant magnitude. Thus, the control objective of inventory loops should consider variations not only in the controlled variable but also in the manipulated variable. Furthermore, inventory loops often have several important constraints associated with both the controlled and manipulated variables. This feature of the inventory loop often necessitates an optimal control strategy with constraint handling. However, since most inventory loops make use of a simple PI controller, constrained optimal control is rarely implemented in the inventory loops. In this study, an analytical design method for PI controllers is developed for optimal regulatory control with explicitly handling the major specifications in the inventory loop.

2. Liquid level control dynamics

The liquid level control system presented in Fig. 1 is described by the following differential equation with the nomenclature of

* Corresponding author. Tel.: +82 53 810 2512; fax: +82 53 811 3262.
E-mail address: mynlee@yu.ac.kr (M. Lee).

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Nomenclature

Roman letters

A	cross-sectional area of tank
H	liquid level deviation
ΔH	level transmitter span
K_L	proportional gain
K_c	dimensionless proportional gain
Q_i	inlet flow rate
Q_o	outlet flow rate
Q_{omax}	maximum outlet flow rate through the control valve
ΔQ_i	magnitude of inlet flow rate change
Q'_o	rate of change of the outlet flow rate
Q'_{omax}	maximum allowable rate of change of the outlet flow rate
t	time

Greek letters

ζ	damping factor
ζ_{min}	minimum allowable damping factor
ζ_{max}	maximum allowable damping factor
τ_I	integral time constant
τ_H	A/K_L
τ_{Hmin}	minimum allowable τ_H
τ_v	hold-up time of tank $(\Delta H)A/Q_{o\,\max}$
ω	weighting factor, $0<\omega<1$
Φ	objective function or performance measure for optimal control

Superscripts

set	set-point
†	extreme point of the objective function
*and**	optimum point on the constraint $\zeta=\zeta_{\min}$ and $\zeta=\zeta_{\max}$, respectively

Acronyms

P	proportional-only control
PI	proportional-integral control
PL	proportional-lag control
DR	decay ratio
$DR_{\max}$	maximum allowable decay ratio

Cheung and Luyben (1979):

$$A\frac{\mathrm{d}H}{\mathrm{d}t} = Q_i - Q_o \tag{1}$$

where A is the constant cross-sectional area of the tank. The liquid level is controlled by adjusting the outlet flow rate and is disturbed by the inlet flow rate.

The Laplace transform of (1) gives

$$H(s) = \frac{1}{As}Q_i(s) - \frac{1}{As}Q_o(s) \tag{2}$$

The liquid level system is a well-known integrating process and a P-only controller leads to no steady-state offset in the case of a servo problem. However, a PI controller is normally used in liquid level loops, since a P-only controller gives a steady-state offset in the case of a regulatory problem. In particular, the modified PI controller in (3) is widely accepted in process industries to avoid the proportional kick for a step set-point change.

$$Q_o(s) = K_L H(s) + \frac{K_L}{\tau_I s}(H(s) - H^{set}(s)) \tag{3}$$

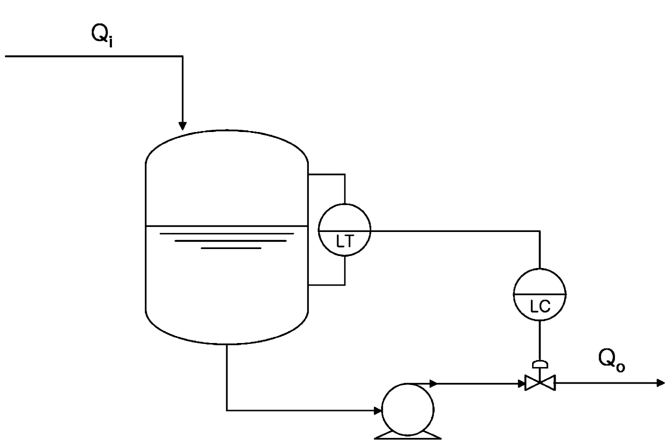

Fig. 1. Schematic representation of a level control loop featuring manipulation of the outlet stream.

where

$$K_L = K_c\frac{Q_{o\,max}}{\Delta H}. \tag{4}$$

Note that the proportional term acts on the process value itself and the integral term acts on the error. This type of a modified PI controller is available in most industrial DCS systems, such as the PID controller with the type-C equation used in the Honeywell™ TDC system. For the modified PI controller, the following closed-loop transfer functions are obtained for the level control system:

$$H(s) = \frac{\tau_H\tau_I}{A}\left(\frac{s}{\tau_H\tau_I s^2 + \tau_I s + 1}\right)Q_i(s) + \left(\frac{1}{\tau_H\tau_I s^2 + \tau_I s + 1}\right)H^{set}(s) \tag{5}$$

$$Q_o(s) = \left(\frac{\tau_I s + 1}{\tau_H\tau_I s^2 + \tau_I s + 1}\right)Q_i(s) - \left(\frac{K_L\tau_H s}{\tau_H\tau_I s^2 + \tau_I s + 1}\right)H^{set}(s) \tag{6}$$

where

$$\tau_H = \frac{A}{K_L} = \frac{\tau_V}{K_c} \tag{7}$$

and

$$\tau_V = \frac{(\Delta H)A}{Q_{o\,max}} \tag{8}$$

The closed-loop characteristic equation is

$$(\tau_I\tau_H)s^2 + \tau_I s + 1 = 0 \tag{9}$$

The tuning parameters K_c and τ_I determine the location of the two poles along with the damping factor that is expressed as follows:

$$\zeta = \frac{1}{2}\sqrt{\frac{\tau_I}{\tau_H}} = \frac{1}{2}\sqrt{\frac{\tau_I K_c}{\tau_V}} \tag{10}$$

3. Formulation of optimal regulatory control

Regulatory control in response to load changes is the major concern in inventory control systems. The PI controller for a liquid level loop is often required to have a smoother, less aggressive control action, even at the expense of less error minimization, which means that the performance measure of the control system must include minimizing not only the error in the controlled variable but also the rate of change of the manipulated variable. At the same time, the controller should be designed to meet all or some of the following typical specifications or constraints in the level loop: (i) the rate of change of the outlet flow should be under a maximum allowable limit, (ii) the decay ratio in the response should be under a maximum allowable limit to avoid a severe oscillatory response, and (iii) the damping coefficient should also be less than a maximum allowable limit in order to secure a suppression speed required.

Based on the operational goal and the three constraints listed above, the optimal design problem of the PI controller in the inventory loop can be defined as finding the controller parameters that minimize the performance measure in (11-1), subject to the constraints in (11-2)–(11-4)

$$\min \Phi = \omega \int_0^\infty \left[\frac{H(t)}{\Delta H}\right]^2 \mathrm{d}t + (1-\omega)\int_0^\infty \left[\frac{Q_o'(t)}{Q'_{o\,max}}\right]^2 \mathrm{d}t \tag{11-1}$$

subject to

$$|Q_o'(t)| \leqslant Q'_{o\,max} \tag{11-2}$$

$$DR \leqslant DR_{max} \tag{11-3}$$

$$\zeta \leqslant \zeta_{max}. \tag{11-4}$$

The first and second terms in (11-1) describe the normalized deviation of the liquid level and the normalized rate of change of the outlet flow, respectively.

3.1. Optimal solution for unconstrained case

Throughout this study, the regulatory problem to a step change in the inlet flow rate (i.e., $Q_i(s) = \Delta Q_i/s$) is considered. By performing certain mathematical manipulations, the objective function Φ given in (11-1) can be expressed in terms of τ_H and ζ as follows (see Appendix A for details):

$$\Phi(\tau_H, \zeta) = 2\omega\left(\frac{\Delta Q_i}{A\Delta H}\right)^2 \tau_H^3\zeta^2 + (1-\omega)\left(\frac{\Delta Q_i}{Q'_{o\,max}}\right)^2 \times \frac{1}{2\tau_H}\left(1+\frac{1}{4\zeta^2}\right) \tag{12}$$

Thus, the unconstrained optimality conditions, i.e., the extremum in the absence of constraints, can be found by solving the following equations simultaneously:

$$\frac{\partial\Phi}{\partial\tau_H} = 6\omega\left(\frac{\Delta Q_i}{A\Delta H}\right)^2 \tau_H^2\zeta^2 - \left(\frac{1-\omega}{2}\right)\left(\frac{\Delta Q_i}{Q'_{o\,max}}\right)^2 \times \left(\frac{1}{\tau_H^2}\right)\left(1+\frac{1}{4\zeta^2}\right) = 0 \tag{13}$$

$$\frac{\partial\Phi}{\partial\zeta} = 4\omega\left(\frac{\Delta Q_i}{A\Delta H}\right)^2 \tau_H^3\zeta - \left(\frac{1-\omega}{4}\right)\left(\frac{\Delta Q_i}{Q'_{o\,max}}\right)^2\left(\frac{1}{\tau_H\zeta^3}\right) = 0 \tag{14}$$

Multiplying (14) by τ_H/ζ gives

$$(\tau_H\zeta)^4 = \left(\frac{1-\omega}{16\omega}\right)\left(\frac{A\Delta H}{Q'_{o\,max}}\right)^2 \tag{15}$$

The optimal τ_H for the unconstrained case can then be obtained by substituting (15) into (13) and rearranging to give

$$\tau_H^{\dagger 2} = \frac{A\Delta H}{2Q'_{o\,max}}\sqrt{\frac{1-\omega}{\omega}} \tag{16}$$

where the superscript † denotes the optimum in the unconstrained case. The optimal ζ can also be obtained as

$$\zeta^\dagger = \frac{1}{\sqrt{2}} \tag{17}$$

It is noted that the optimal value of the damping coefficient is equal to $\frac{1}{\sqrt{2}}$ and is independent of the process dynamics and weighting factor. The optimal tuning values of $K_c^\dagger$ and $\tau_I^\dagger$ for the unconstrained case can be simply calculated using (7) and (10).

The following simple relation can also be derived for the optimal tuning of the unconstrained case:

$$K_c^\dagger \tau_I^\dagger = 2\tau_V. \tag{18}$$

Note that product of $K_c^\dagger$ and $\tau_I^\dagger$ is proportional to the hold-up time of a level tank. Seki and Ogawa (1998) also came to the same conclusion as that described by (17) and (18) for the optimal control of a level loop.

Remark 1. For the optimal tuning, the product of $K_c^\dagger$ and $\tau_I^\dagger$ should be kept constant by doubling the hold-up time, regardless of the weighting factor. When one of the PI parameters, $K_c^\dagger$ and $\tau_I^\dagger$, is changed during tuning, the other parameter should also be adjusted so that the product remains constant.

3.2. Optimal solution for constrained case

When the constraints given in (11-2)–(11-4) are considered in the controller design, the global optimum can be located either on the extreme point of the objective function or on the constraint boundary. Note that an extreme point denotes an extremum in the absence of constraints. To deal with the constrained cases, all of the constraints have to be expressed in terms of the independent variables, τ_H and ζ. The rate of change of the outlet flow rate for a step disturbance can be determined by using the properties of the Laplace transform of the derivative of the outlet flow, which is given by

$$L\left[\frac{\mathrm{d}Q_o(t)}{\mathrm{d}t}\right] = sQ_o(s) - Q_o(t)_{t=0} \tag{19}$$

where $Q_o(t)_{t=0} = 0$.

Throughout this study, it is assumed that the controller is designed to give a response with $\zeta \geqslant 0.5$ in order to avoid a severe oscillatory response. In this case, since the largest value of the rate of change occurs at $t = 0$, by applying the initial value theorem to (6), it is easily determined that

$$\left[\frac{\mathrm{d}Q_o(t)}{\mathrm{d}t}\right]_{t=0} = \frac{\Delta Q_i}{\tau_H} \tag{20}$$

Therefore, the constraint given in (11-2) can be expressed in terms of τ_H:

$$\tau_H \geqslant \frac{\Delta Q_i}{Q'_{o\,max}} \equiv \tau_{H\,min} \tag{21}$$

Furthermore, using the relation between the decay ratio and the damping coefficient in the second-order process, the constraint imposed by (11-3) can be converted to the following inequality condition in terms of ζ:

$$\zeta \geqslant \frac{1}{\sqrt{1+(2\pi/\ln DR_{\max})^2}} \equiv \zeta_{\min} \tag{22}$$

Note that since all of the constraints are linear, the solution region is convex. The feasibility of the solution region should be firstly checked for a given constraint set, which can be easily done from (11-4) and (22).

Once the constraints applied are confirmed as giving a feasible solution region, the next step is to determine the global optimal condition. The global optimum can exist either on the extreme point of the objective function or on the boundary of a constraint. The local optimum value of ζ on the boundary of $\tau_H = \tau_{Hmin}$ can be calculated from (14) by replacing τ_H with τ_{Hmin}:

$$\zeta^* = \frac{1}{\sqrt{2}}\left(\frac{1}{\tau_{H\ min}}\right)\left(\frac{1-\omega}{4\omega}\right)^{1/4}\sqrt{\frac{A\Delta H}{Q'_{o\ max}}} = \frac{1}{\sqrt{2}}\frac{\tau_H^\dagger}{\tau_{H\ min}} \tag{23}$$

The local optimum value of τ_H on the boundary of $\zeta = \zeta_{min}$ can be found from (13) by replacing ζ with ζ_{min}:

$$\tau_H^* = \left(\frac{4\zeta_{min}^2+1}{12\zeta_{min}^4}\right)^{1/4}\left(\frac{1-\omega}{4\omega}\right)^{1/4}\sqrt{\frac{A\Delta H}{Q'_{o\ max}}} = \left(\frac{4\zeta_{min}^2+1}{12\zeta_{min}^4}\right)^{1/4}\tau_H^\dagger \tag{24}$$

Similarly, the local optimum value of τ_H on the boundary of $\zeta = \zeta_{\max}$ can be obtained as

$$\tau_H^{**} = \left(\frac{4\zeta_{max}^2+1}{12\zeta_{max}^4}\right)^{1/4}\left(\frac{1-\omega}{4\omega}\right)^{1/4}\sqrt{\frac{A\Delta H}{Q'_{o\ max}}} = \left(\frac{4\zeta_{max}^2+1}{12\zeta_{max}^4}\right)^{1/4}\tau_H^\dagger \tag{25}$$

It is clear from (23) that the local optimum ζ^* is always less than $\frac{1}{\sqrt{2}}$ (or $\zeta^\dagger$) when $\tau_H{}^\dagger < \tau_{Hmin}$. Furthermore, since $g(\zeta) = (4\zeta^2+1/12\zeta^4)^{1/4}$ is a monotonically decreasing function in ζ with $g(1/\sqrt{2}) = 1$, $\tau_H{}^*$ is less than $\tau_H{}^\dagger$ for $\zeta^\dagger < \zeta_{min}$ and $\tau_H{}^{**}$ is greater than $\tau_H{}^\dagger$ for $\zeta^\dagger > \zeta_{max}$. These relationships indicate that the contours of the objective function are left skewed in the ζ–τ_H plane.

Based on the contour characteristics, the condition and location of the global optimums of (ζ,τ_H) for every possible case are found as listed in Table 1. Fig. 2 shows five possible cases of a global optimum with the contours of the objective function and the constraints imposed by (11-2)–(11-4). Once the global optimum $(\zeta^{opt},\tau_H{}^{opt})$ is obtained, the corresponding optimal PI parameters can be simply calculated from (7), (8), and (10) as

$$K_C^{opt} = \frac{(\Delta H)A}{Q_{o\ max}\tau_H^{opt}} \tag{26}$$

Table 1
Global optimums of (ξ,τ_H) for the constrained case

Case	Condition	Location	Solution
A	$(\tau_H{}^\dagger \geqslant \tau_{H\ min}$ and $\zeta_{max} \geqslant \zeta^\dagger \geqslant \zeta_{min})$	At the extremum $(\zeta^\dagger\ \tau_H{}^\dagger)$	$(\zeta^\dagger\ \tau_H{}^\dagger)$
B	$(\tau_H{}^\dagger < \tau_{H\ min}$ and $\zeta_{max} \geqslant \zeta^* \geqslant \zeta_{min})$ or $(\zeta^\dagger > \zeta_{max}$ and $\tau_H{}^{**} < \tau_{H\ min})$	On the constraint $\tau_H = \tau_{H\ min}$	$(\zeta^*,\tau_{H\ min})$
C	$(\zeta^\dagger > \zeta_{min}$ and $\tau_H{}^* \geqslant \tau_{H\ min})$	On the constraint $\zeta = \zeta_{min}$	$(\zeta_{min},\tau_H{}^*)$
D	$(\zeta^\dagger > \zeta_{max}$ and $\tau_H{}^{**} \geqslant \tau_{H\ min})$ or $(\tau_H{}^\dagger < \tau_{H\ min}$ and $\zeta^* > \zeta_{max})$	On the constraint $\zeta = \zeta_{max}$	$(\zeta_{max},\tau_H{}^{**})$
E	$(\tau_H{}^\dagger < \tau_{H\ min}$ and $\zeta^* < \zeta_{min})$ or $(\zeta^\dagger < \zeta_{min}$ and $\tau_H{}^* < \tau_{H\ min})$	On the vertex by $\tau_H = \tau_{H\ min}$ and $\zeta = \zeta_{min}$	$(\zeta_{min},\tau_{H\ min})$

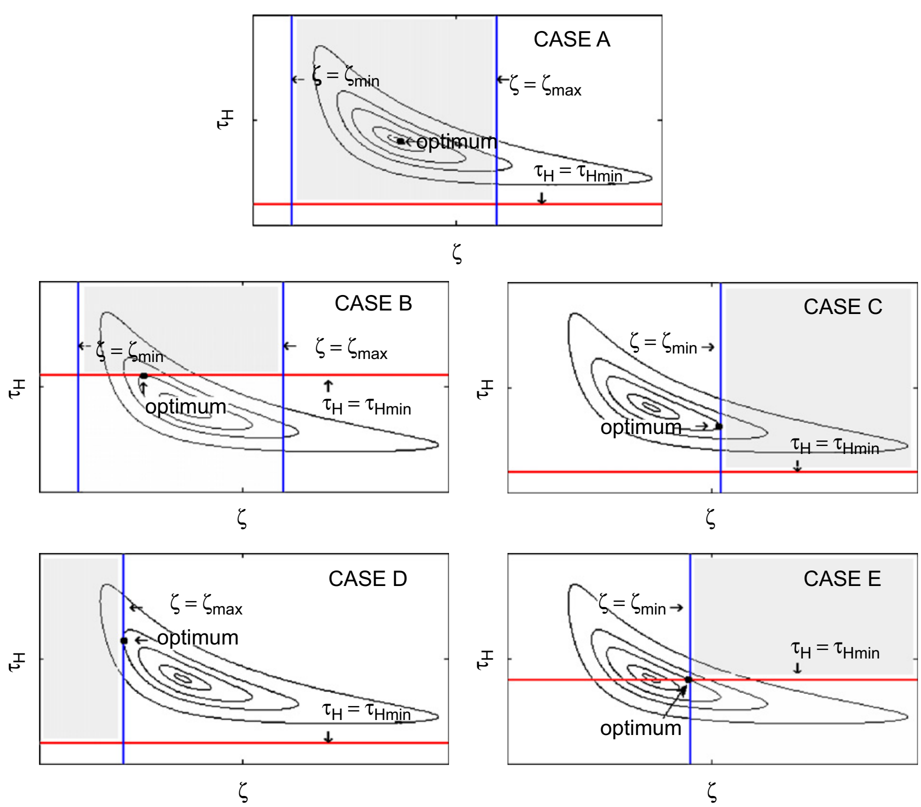

Fig. 2. Possible cases of a global optimum location: the shaded region denotes a feasible region.

$$\tau_I^{opt} = 4(\zeta^{opt})^2 \tau_H^{opt} \quad (27)$$

4. Illustrative examples

Consider a liquid level system as follows: the liquid level of a tank with a cross-section area of $1\,m^2$ and a working volume ($A\Delta H$) of $2\,m^3$ is controlled by a PI controller. The maximum outlet flow (Q_{omax}) is $4\,m^3$/min. The initial steady-state level is 50% and the nominal flow rates of the inlet and outlet are both $1\,m^3$/min. The maximum expected change in the inlet flow (ΔQ_i) is $1\,m^3$/min. The maximum allowable rate of change of the outlet flow (Q'_{omax}) is $1.5\,m^3/min^2$.

Example 1. Optimal tuning: unconstrained case.

Suppose that there is no hard control specification. Optimal tuning can then be calculated based on (16) and (17). For example, if $\omega = 0.5$, $\tau_H^\dagger = 1/\sqrt{3}$ and $\zeta^\dagger = 1/\sqrt{2}$ are obtained. The optimal PI parameters are calculated using (26) and (27) as $K_c = \sqrt{3}/2$ and $\tau_I = 2/\sqrt{3}$ min.

The responses in the level and outlet flow rate for the proposed optimal tuning with various weighting factors are shown in Fig. 3. In the simulation, a step change of $1\,m^3$/min in the inlet flow rate is introduced at 5 min and sequentially the level set-point undergoes a 25% step increase at 20 min. As seen in the figure, the lower the weighting factor, ω, the more slowly the outlet flow changes, but the higher the peak level becomes and the more sluggishly the level is controlled. As ω is increased, a smaller peak can be obtained at the cost of a higher rate of change in the outlet flow rate. In this manner, a clear tradeoff can be achieved between the tightness in the level control and the smoothness in the outlet flow change with only the single tuning parameter, ω.

To confirm the advantage of the proposed method, the closed loop performance provided by the proposed PI controller is compared with that afforded by the IMC-PI tuning method (Rivera et al., 1986). The weighting factor ω is set to 0.5 in the proposed method. In order to provide a fair comparison, the closed loop time constant in the IMC-PI method is adjusted so that both controllers yield the same maximum peak level. As shown in Fig. 4, the PI controller using the proposed method gives a smaller maximum rate of change of the outlet flow, as well as a faster settling time in the level response. The value of the performance measure in (11-1) was also evaluated for each method, and the proposed PI controller gave a smaller value of 0.2988 than that given by the IMC-PI tuning method of 0.3580.

Example 2. Optimal tuning: constrained case.

When the control specifications given by (11-2)–(11-4) have to be strictly satisfied, the optimal tuning values can be obtained by categorizing the global optimum case from Table 1. Suppose that the weighting factor is set to $\omega = 0.8$ for the liquid level system above. As an illustrative example, consider cases I, II, and III, where the maximum allowable decay ratios are 0.0005, 0.1, and 0.1 and the maximum allowable damping coefficients are 1.0, 1.0, and 0.4, respectively, while the maximum allowable rate of change of the outlet flow is $1.5\,m^3/min^2$ in all three cases. From (21), $\tau_{H\,min} = 0.6667$ is obtained. The minimum allowable damping coefficients corresponding to the maximum decay ratios, calculated using (22), are 0.7708 for case I and 0.3441 for cases II and III. Therefore, cases I, II, and III correspond to cases E, B, and D in Table 1 and Fig. 2, respectively. The optimal PI parameters for the three cases are $K_c = 0.75$, 0.75, and 0.5697 and $\tau_I = 1.584$, 1.0, and 0.562, respectively.

Fig. 5 compares the responses for the level and rate of change of the outlet flow for the three cases. In the simulation, a step change of $1\,m^3$/min in the inlet flow rate is introduced at 1 min. The global optimum is located at the vertex point 숀 by the two constraints $\zeta = \zeta_{min}$ and $\tau_H = \tau_{H\,min}$ for case I, on the constraint $\tau_H = \tau_{H\,min}$ for case II, and on the constraint $\zeta = \zeta_{max}$ for case III. As seen in the figure, all of the responses strictly satisfy the given control specifications.

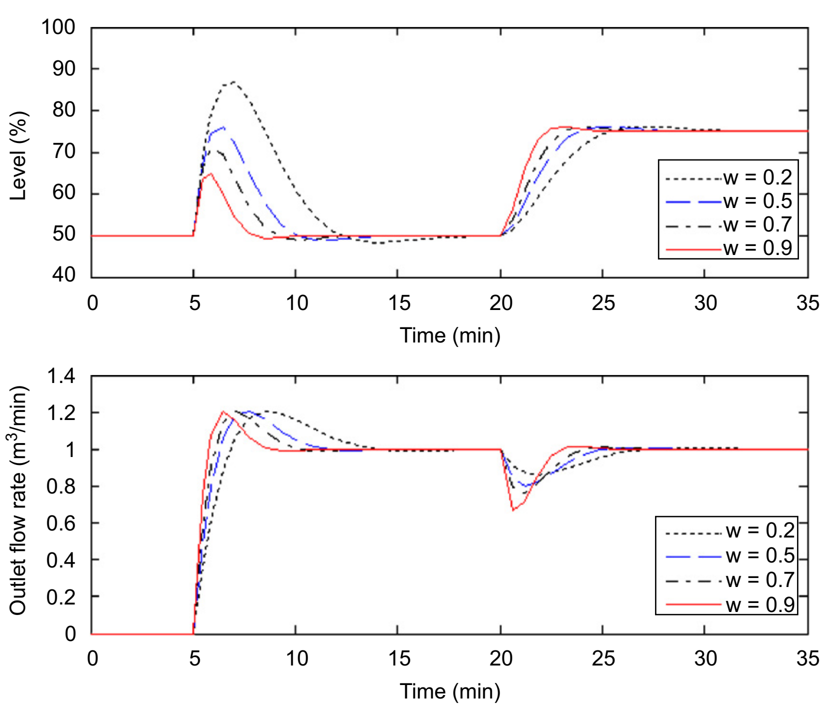

Fig. 3. Level and outlet flow responses using the proposed PI controller (unconstrained case).

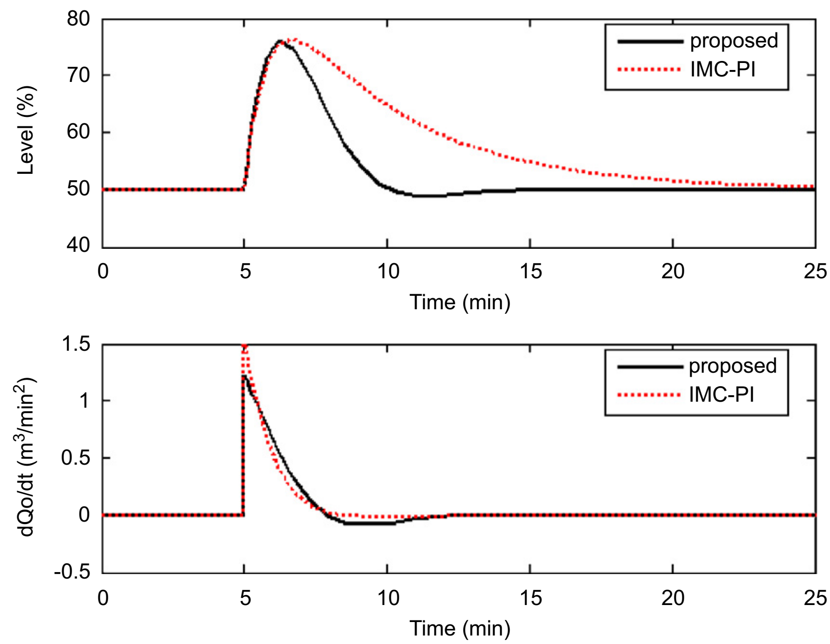

Fig. 4. Comparison of responses by the proposed method and the IMC-PI tuning method (unconstrained case, $w = 0.5$).

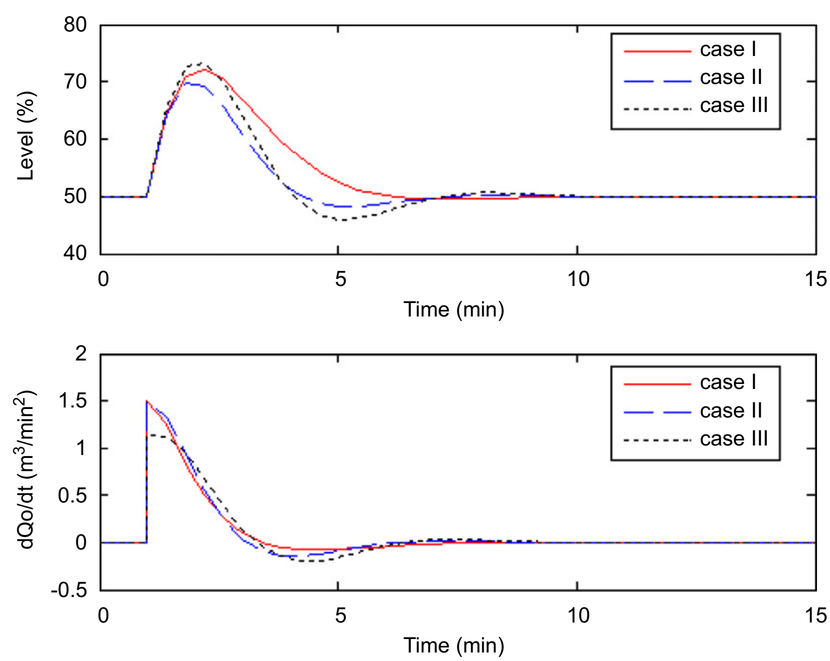

Fig. 5. Level and rate of change of outlet flow responses using the proposed PI controller (constrained case): case I ($DR_{max} = 0.0005$, $\zeta_{max} = 1.0$), case II ($DR_{max} = 0.1$, $\zeta_{max} = 1.0$), case III ($DR_{max} = 0.1$, $\zeta_{max} = 0.4$).

Example 3. Robustness against modeling error in dead time and area.

Level processes generally do not have a dead time except for certain rare occasions such as level loops in which the control valve has a dead zone. Therefore, the dead time in level processes cannot be too large and has to be maintained at almost 10% of the hold-up time (Wu et al., 2001).

Fig. 6 shows the results assuming that the level process in Example 1 has a dead time of 20% of the hold-up time. The weighting factor is set to 0.5. The issue of robustness is also studied by examining $\pm 20\%$ errors in the cross-sectional area and the results are presented in Fig. 6. The responses shown in the figure indicate that the realistic uncertainties in the dead time and area have little effect on the control performance.

5. Conclusions

An analytical design method for the optimal control of a liquid level loop is developed by solving a constrained optimization problem. One of the main drawbacks of simple PID controllers is that they cannot handle the constraints explicitly. The proposed design method explicitly deals with the important constraints in the inventory loop, as well as minimizing the optimal control specification. Several examples were presented to illustrate the

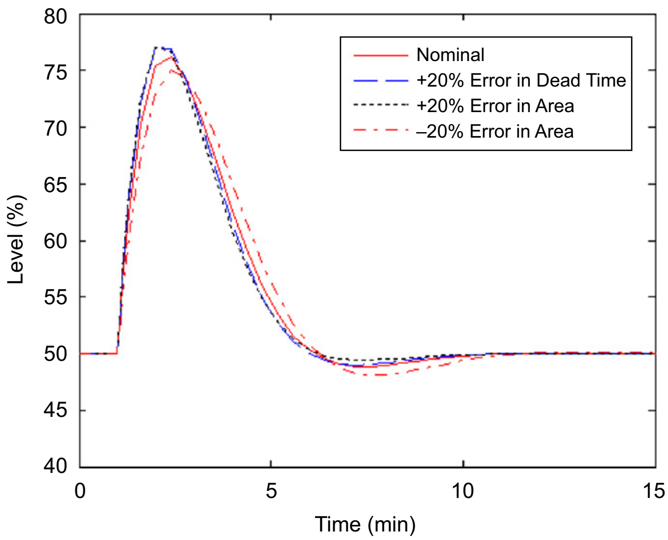

Fig. 6. Effect of modeling error in dead time and area on the control performance.

effectiveness of the proposed design method. The results also showed that the proposed method gives satisfactory responses, not only in the nominal condition but also under uncertainties in the dead time and cross-sectional area.

Acknowledgment

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Appendix A. Derivation of the objective function Φ in (12)

From (5) with $Q_i(s)=\Delta Q_i/s$ and $H^{set}(s)=0$, the liquid level response is obtained as

$$H(t)=\frac{\Delta Q_i}{A}\left(\frac{e^{r_1 t}-e^{r_2 t}}{r_1-r_2}\right)\quad \text{for } r_1\neq r_2 \tag{A1}$$

where r_1 and r_2 are the roots of the characteristic equation $s^2+(1/\tau_H)s+(1/\tau_I\tau_H)=0$.

Thus,

$$r_1 r_2=\frac{1}{\tau_I\tau_H} \tag{A2}$$

$$r_1+r_2=-\frac{1}{\tau_H} \tag{A3}$$

$$r_1-r_2=\frac{1}{\tau_H}\sqrt{1-\frac{4\tau_H}{\tau_I}}=\frac{1}{\tau_H}\sqrt{\frac{\zeta^2-1}{\zeta^2}} \tag{A4}$$

Therefore, the integral square error of the normalized liquid level in (11-1) becomes

$$\begin{aligned}\omega\int_0^\infty\left[\frac{H(t)}{\Delta H}\right]^2 dt &= \omega\left(\frac{\Delta Q_i}{A\Delta H}\right)^2\left(\frac{1}{r_1-r_2}\right)^2\left[\frac{2}{r_1+r_2}-\frac{1}{2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right]\\ &= 2\omega\left(\frac{\Delta Q_i}{A\Delta H}\right)^2\tau_H^3\zeta^2\end{aligned} \tag{A5}$$

Now, consider the integral square error of the normalized rate of change of the outlet flow rate in (11-1).

From (6), with $Q_i(s)=\Delta Q_i/s$ and $H^{set}(s)=0$, the Laplace transform of the outlet flow rate is

$$Q_o(s)=\left(\frac{\tau_I s+1}{\tau_H\tau_I s^2+\tau_I s+1}\right)\frac{\Delta Q_i}{s} \tag{A6}$$

Therefore, using the property of the Laplace transform for a derivative, i.e., $Q'_o(s)\equiv L[dQ_o(t)/dt]=sQ_o(s)$, the rate of change of the outlet flow rate is obtained by

$$Q'_o(t)=\Delta Q_i\left(\frac{r_2^2 e^{r_2 t}-r_1^2 e^{r_1 t}}{r_1-r_2}\right)\quad \text{for } r_1\neq r_2 \tag{A7}$$

Therefore, the integral square error of the normalized rate of change of the outlet flow rate becomes

$$\begin{aligned}(1-\omega)\int_0^\infty\left[\frac{Q'_o(t)}{Q'_{o\,max}}\right]^2 dt &= (1-\omega)\left(\frac{\Delta Q_i}{Q'_{o\,max}}\right)^2\left(\frac{1}{r_1-r_2}\right)^2\\ &\times\left[-\frac{r_2^3}{2}-\frac{r_1^3}{2}+\frac{2r_1^2 r_2^2}{r_1+r_2}\right]\\ &= (1-\omega)\left(\frac{\Delta Q_i}{Q'_{o\,max}}\right)^2\left(\frac{1}{r_1-r_2}\right)^2\left(-\frac{1}{2}\right)\\ &\times\left[(r_1+r_2)^3-3r_1r_2(r_1+r_2)-\frac{4r_1^2r_2^2}{r_1+r_2}\right]\\ &= (1-\omega)\left(\frac{\Delta Q_i}{Q'_{o\,max}}\right)^2\frac{1}{2\tau_H}\left(1+\frac{1}{4\zeta^2}\right)\end{aligned} \tag{A8}$$

References

Buckley, P. (1983). Recent advances in averaging level control. In *Productivity through control technology* (pp. 18–21), Houston.

Cheung, T., & Luyben, W. (1979). Liquid-level control in single tanks and cascades of tanks with P-only and PI feedback controllers. *Industrial and Engineering Chemistry Fundamentals*, *18*(1), 15–21.

Luyben, W., & Buckley, P. S. (1977). A proportional-lag controller. *Instrumentation Technology*, *24*(12), 65–68.

MacDonald, K., McAvoy, T., & Tits, A. (1986). Optimal averaging level control. *AIChE Journal*, *32*, 75–86.

Marlin, T. E. (1995). *Process control*. Mcgraw-Hill 581.

Rivera, D. E., Morari, M., & Skogestad, S. (1986). Internal model control, 4: PID controller design. *Industrial & Engineering Chemistry Process Design and Development*, *25*(1), 252–265.

Seki, H., & Ogawa, M. (1998). Japan Patent # 2811041.

Wu, K., Yu, C., & Cheung, Y. (2001). A two degree of freedom level control. *Journal of Process Control*, *11*, 311–319.

Yang, D. R., Seborg, D. E., & Mellichamp, D. A. (1994). The influence of inventory control dynamics on distillation composition control. *Control Engineering Practice*, *2*(6), 27–32.