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Constrained Optimal Control of Liquid Level Loop Using a Conventional Proportional-Integral Controller

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An analytical design method for a conventional proportional-integral (PI) controller is developed for the optimal control of the liquid level loop in order to explicitly handle important control specifications. The constrained optimal control problem for the liquid level loop is formulated and converted into an unconstrained optimization problem to find an optimal PI controller tuning rule. The proposed PI controller minimizes the rate of change of the outlet flow and the deviation of the level subject to two important constraints in the level loop: the maximum allowable rate of change in the outlet flow rate and the maximum allowable deviation of the level. The simulation results show that the constrained optimal control of the liquid level loop can be successfully implemented using a simple PI controller with the proposed design method.

Keywords Constrained optimization; Constraint control; Inventory control; Liquid level control; Optimal control; Proportional-integral (PI) controller tuning

Introduction

Liquid level loops are commonly encountered in process industries. Since the desired production rates and inventories are achieved through the proper control of flows and levels, level control is quite important for the successful operation of most chemical plants (Marlin, 1995). The industrial importance of level loops has led to extensive research interest to achieve the enhanced control performance of the level loop.

Luyben and Buckley (1977) proposed a proportional-lag (PL) control as a potentially good solution to the problem of liquid level control systems with feedforward compensation. Cheung and Luyben (1979) studied a tuning procedure of the proportional-integral (PI) controller with a design chart using specifications for the maximum level deviation and maximum rate of change of the manipulated flow. Marlin (1995) proposed a simple tuning method of the PI controller to meet the maximum level deviation only. Rivera et al. (1986) proposed the internal model control (IMC)-based proportional-integral-differential (PID) tuning rule for the critically damped closed-loop response of a liquid level loop. Several tuning methods were presented for averaging level control (MacDonald et al., 1986; St. Clair, 1993).

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Seki and Ogawa (1998) observed that the optimal response in the level loop occurs at a particular value of the damping coefficient. Several nonlinear PI controllers were also discussed with the goal of providing fast control action for large errors and slow action for small errors in liquid level loops (Buckley, 1983). Wu et al. (2001) proposed a two degree of freedom control structure to address both the regulatory and servo problems in level control.

In a level control loop, the manipulated flow is often situated upstream of a critical unit. In this case, controlling the behavior of the outlet flow is as important as controlling that of the liquid level itself, in order to avoid rapid variations with a significant magnitude. Therefore, a level controller is required to provide nonaggressive and smooth control action as well minimizing the deviation of the level. Furthermore, a level loop normally has two important requirements: (1) the rate of change of the outlet flow should be kept below a specified allowable limit, and (2) the deviation of the level should also be within a specified allowable limit. For this reason, level control problems can be considered as a typical constrained optimal control problem. In spite of its industrial and economic importance, the constrained optimal control strategy has rarely been employed in practical level loops. One of the main reasons is that most industrial level loops use simple P-only or PI controllers, which are generally accepted as being too simple to implement any sophisticated control strategy. The main obstacle to achieving optimal control is that it normally requires an optimization package to find the optimal control action, and many practitioners are not familiar with the use of these complicated packages. Using a sophisticated advanced controller such as a model predictive controller might be a solution for the constraint control of the level loop, but it cannot be considered to be a practical approach for many reasons.

In this article, we developed an analytical design method for a conventional PI controller that enables the constrained optimal control of the liquid level loop. The constrained optimal level control problem is first formulated and then converted into a simple form with two independent variables by using a proper variable transformation. The Lagrangian multiplier method is applied to handle the constraint optimization problem, and finally the optimal PI tuning rule is found from the analysis of the global optimum condition. The proposed method is shown to deal with the two major control specifications in the level loops explicitly, while minimizing the optimal control performance measure.

Dynamic Behavior of Liquid Level Loop

The liquid level control system presented in Figure 1 is simply described as

$$A\frac{dH}{dt} = Q_i - Q_o \tag{1}$$

where the liquid level is controlled by adjusting the outlet flow rate and is disturbed by the inlet flow rate.

The Laplace transform of Equation (1) gives

$$H(s) = \frac{1}{As}Q_i(s) - \frac{1}{As}Q_o(s) \tag{2}$$

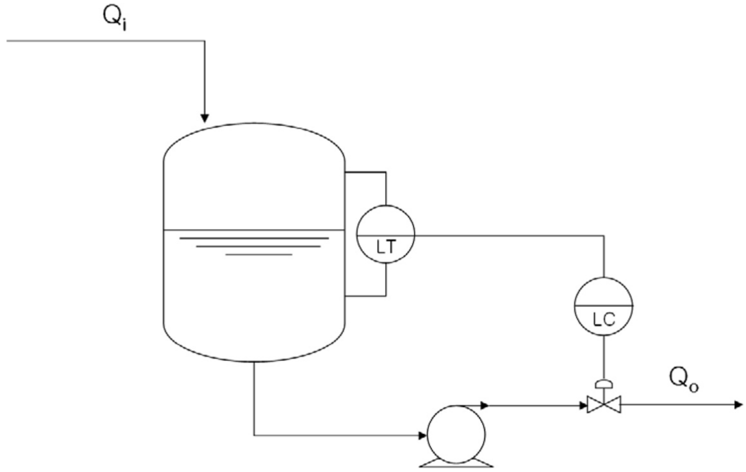

Figure 1. Schematic of a level control loop featuring manipulation of the outlet stream.

If a PI controller is used

$$Q_o(s) = K_L\left(1 + \frac{1}{\tau_I s}\right)(H(s) - H^{set}(s)) \tag{3}$$

where

$$K_L = K_c \frac{Q_{o\max}}{\Delta H} \tag{4}$$

The closed-loop transfer functions for the level control system are then

$$H(\mathrm{s}) = \frac{\tau_H \tau_I}{A} \frac{s}{\tau_H \tau_I s^2 + \tau_I s + 1} Q_i(s) + \frac{\tau_I s + 1}{\tau_H \tau_I s^2 + \tau_I s + 1} H^{set}(s) \tag{5}$$

$$Q_o(s) = \frac{\tau_I s + 1}{\tau_H \tau_I s^2 + \tau_I s + 1} Q_i(s) - \frac{K_L \tau_H s(\tau_I s + 1)}{\tau_H \tau_I s^2 + \tau_I s + 1} H^{set}(s) \tag{6}$$

where

$$\tau_H = \frac{A}{K_L} = \frac{\tau_V}{K_c} \tag{7}$$

and

$$\tau_V = \frac{(\Delta H)A}{Q_{o\max}} \tag{8}$$

The damping factor of the above closed-loop characteristic equation is expressed as

$$\zeta = \frac{1}{2}\sqrt{\frac{\tau_I}{\tau_H}} \tag{9}$$

Formulation of Constrained Optimal Control

Since the main disturbance in the level loop is the variation of the inlet flow, regulatory control against a load change is the major concern in the level control. The control objective is to minimize both the rate of change of the outlet flow and the deviation of the level against a load variation, while keeping both (1) the rate of change of the outlet flow within a specified allowable limit and (2) the deviation of the level within a specified allowable limit.

Therefore, the constrained optimal control problem of the level loop can be defined as finding the controller parameters that minimize the performance measure in Equation (10a) for a given step change in Q_i, subject to the constraints in Equations (10b) and (10c):

$$\min \Phi = \omega \int_0^\infty \left[\frac{H(t)}{\Delta H}\right]^2 dt + (1-\omega)\int_0^\infty \left[\frac{Q'_o(t)}{Q'_{o\ \max}}\right]^2 dt \tag{10a}$$

subject to

$$\left|Q'_o(t)\right| \leq Q'_{o\ \max} \tag{10b}$$

$$|H(t)| \leq H_{\max} \tag{10c}$$

where each performance term in Equation (10a) is normalized to a dimensionless one.

Throughout this study, we consider the regulatory problem with regard to a step change in the inlet flow, i.e., $Q_i(s) = \Delta Q_i/s$. By performing certain mathematical manipulations, the constrained optimal control problem in Equations (10a)–(10c) can be expressed in terms of τ_{H} and ζ as follows (see Appendix for details):

$$\min \Phi(\tau_{\mathrm{H}}, \zeta) = \alpha\tau_{\mathrm{H}}^3\zeta^2 + \beta \cdot \frac{1}{\tau_{\mathrm{H}}}\left(1 + \frac{1}{4\zeta^2}\right) \tag{11a}$$

subject to

$$\tau_{\mathrm{H}} \geq \gamma_{\mathrm{L}} h(\zeta) \tag{11b}$$

$$\tau_{\mathrm{H}} \leq \frac{\gamma_{\mathrm{U}}}{g(\zeta)} \tag{11c}$$

where

$$\alpha = 2\omega\left(\frac{\Delta Q_i}{A\Delta H}\right)^2,\ \beta = \left(\frac{1-\omega}{2}\right)\left(\frac{\Delta Q_i}{Q'_{o\max}}\right)^2,\ \gamma_{\mathrm{U}} = \frac{AH_{\max}}{\Delta Q_i},\ \text{and}\ \gamma_{\mathrm{L}} = \frac{\Delta Q_i}{Q'_{o\max}} \tag{12}$$

$h(\zeta)$ and $g(\zeta)$ are given by Equations (A11) and (A15) in the Appendix, respectively.

Condition for Feasible $Q'_{o\max}$ and $H_{\max}$ Specifications

It is preferable for a level controller to allow tight $Q'_{o\max}$ and $H_{\max}$ specifications. However, since the feasible region is surrounded by the two constraints in Equations (11b) and (11c), as smaller $Q'_{o\max}$ and/or $H_{\max}$ specifications are applied, the feasible region bounded by these constraints also becomes smaller and eventually unavailable. Figure 2 shows the effect of $Q'_{o\max}$ and $H_{\max}$ specifications on the constraints and feasible region for the liquid level system. As seen from the figure, as $Q'_{o\max}$ (or $H_{\max}$) specification decreases for a given $H_{\max}$ (or $Q'_{o\max}$) specification, the feasible region becomes narrow, then available only at the tangent point, and finally unavailable. It is clear that for a given $Q'_{o\max}$ (or $H_{\max}$) specification, there is a minimum achievable limit in the $H_{\max}$ (or $Q'_{o\max}$) specification, and it occurs at the tangent point.

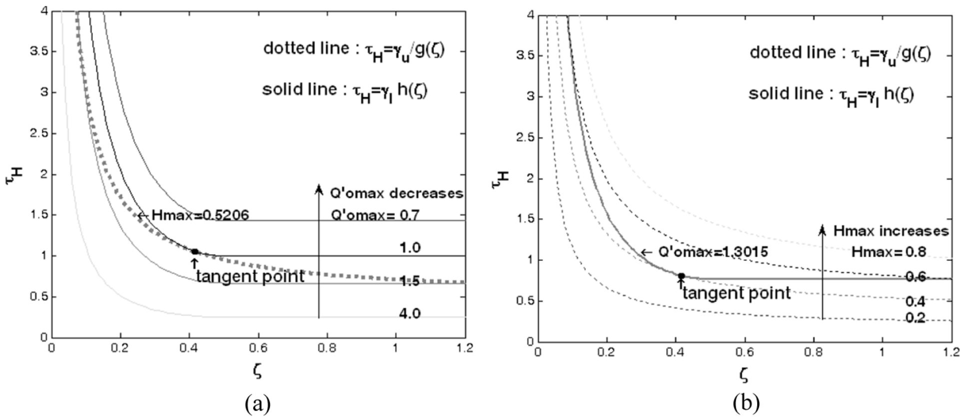

Figure 2. Effect of $Q'_{o\max}$ (a) and $H_{\max}$ (b) specifications on the constraints and feasible region.

The tangent point can be found from Equations (11b) and (11c) by using the property that the intersections of the two constraints satisfies

$$h(\zeta)g(\zeta) = \frac{H_{\max} Q'_{o\max}}{(\Delta Q_i^2/A)} \tag{13}$$

The above equation indicates that the point of tangency by the two constraints occurs at $dh(\zeta)g(\zeta)/d\zeta = 0$. Thus, the ζ value at the tangent point can be obtained by differentiating Equation (13) with respect to ζ and setting the derivative to zero to get $\zeta^t = 0.4040$. Therefore, for any $Q'_{o\max}$ (or $H_{\max}$) specification, the tightest $H_{\max}$ (or $Q'_{o\max}$) specification always occurs at $\zeta^t = 0.4040$ and thus can be calculated by

$$H_{\max} Q'_{o\max} = h(\zeta^t)g(\zeta^t)(\Delta Q_i^2/A) = 0.5206(\Delta Q_i^2/A) \tag{14}$$

Remark. For a given $(Q'_{o\max}, H_{\max})$ specification, if $H_{\max} Q'_{o\max} \geq 0.5206(\Delta Q_i^2/A)$, then the specification is feasible. On the other hand, if $H_{\max} Q'_{o\max} < 0.5206(\Delta Q_i^2/A)$, then the specification is infeasible.

Controller Design for Constrained Optimal Control

Applying the Lagrangian multiplier (Dennis, 1959) with slack variables converts the constrained optimization problem in Equations (11a)–(11c) into an unconstrained equivalent problem with augmented objective function, as follows:

$$\begin{aligned} &\min L(\tau_H, \zeta, \varpi_1, \varpi_2, \sigma_1, \sigma_2) \\ &= \alpha \tau_H^3 \zeta^2 + \frac{\beta}{\tau_H}\left(1 + \frac{1}{4\zeta^2}\right) + \varpi_1(\tau_H - \gamma_L h(\zeta) - \sigma_1^2) + \varpi_2\left(\frac{\gamma_U}{\tau_H} - g(\zeta) - \sigma_2^2\right) \end{aligned} \tag{15}$$

where ϖ_i is a Lagrange multiplier ($\varpi_i \leq 0$), and σ_i is a slack variable.

The necessary condition for an optimum solution of Equation (15) is

$$\frac{\partial L}{\partial \tau_H} = 3\alpha \tau_H^2 \zeta^2 - \frac{\beta}{\tau_H^2}\left(1 + \frac{1}{4\zeta^2}\right) + \varpi_1 - \frac{\varpi_2 \gamma_U}{\tau_H^2} = 0 \tag{16a}$$

$$\frac{\partial L}{\partial \zeta} = 2\alpha\tau_{\mathrm{H}}^3\zeta - \frac{\beta}{2\tau_{\mathrm{H}}}\frac{1}{\zeta^3} - \varpi_2 g'(\zeta) = 0 \tag{16b}$$

$$\frac{\partial L}{\partial \varpi_1} = \tau_{\mathrm{H}} - \gamma_{\mathrm{L}} h(\zeta) - \sigma_1^2 = 0 \tag{16c}$$

$$\frac{\partial L}{\partial \varpi_2} = \frac{\gamma_{\mathrm{U}}}{\tau_{\mathrm{H}}} - g(\zeta) - \sigma_2^2 = 0 \tag{16d}$$

$$\frac{\partial L}{\partial \sigma_1} = -2\varpi_1\sigma_1 = 0 \tag{16e}$$

$$\frac{\partial L}{\partial \sigma_2} = -2\varpi_2\sigma_2 = 0 \tag{16f}$$

Figure 3 shows several possible instances of a global optimum with the contours of the objective function and the constraints given in Equations (11a)–(11c). Four cases are possible with respect to the global optimum location: (1) the global optimum is in the interior of the constraint set; (2) the global optimum is on the constraint $\tau_{\mathrm{H}} = \gamma_{\mathrm{L}} h(\zeta)$; (3) the global optimum is on the constraint $\tau_{\mathrm{H}} = \gamma_{\mathrm{U}}/g(\zeta)$; and (4) the global optimum is located on the vertex point formed by the two constraints.

Based on this property for a global optimum location, the global optimum of $(\zeta, \tau_{\mathrm{H}})$ for each possible case is found as follows.

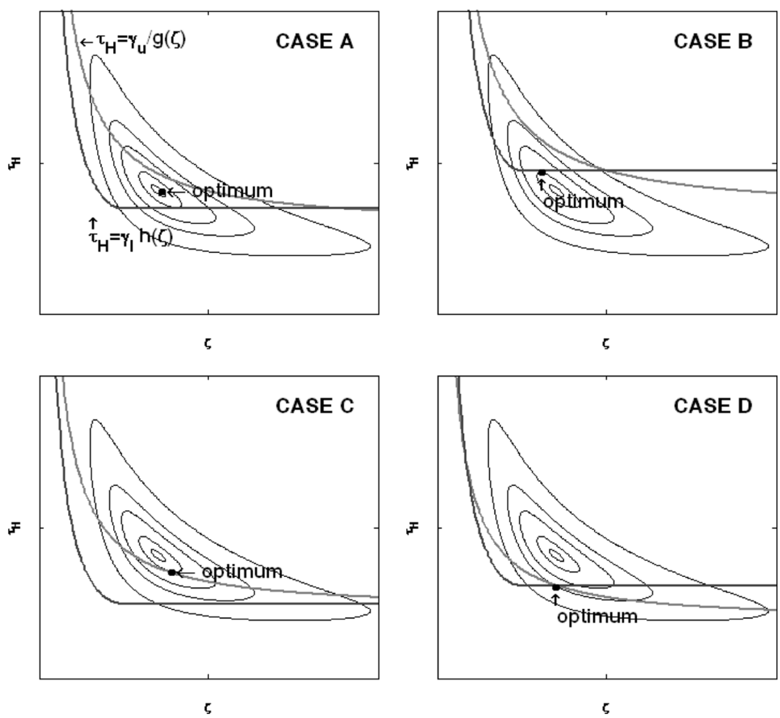

Figure 3. Typical contours and constraints with four possible cases of optimum location.

When a Feasible Region is Bounded ($\gamma_L \geq \gamma_U$)

Since $\gamma_{\mathrm{U}}/g(\zeta)$ and $\gamma_{\mathrm{L}}h(\zeta)$ approach γ_{U} and γ_{L}, respectively, as ζ increases, the feasible region is bounded when $\gamma_{\mathrm{L}} \geq \gamma_{\mathrm{U}}$. The case of the bounded feasible region is likely to occur when the $Q'_{o\max}$ and/or $H_{\max}$ specifications are tight. The following four cases are possible with respect to the location of a global optimum.

Case A ($\varpi_1 = \varpi_2 = 0$). The extreme point $(\zeta^\dagger, \tau_{\mathrm{H}}^\dagger)$ is in the interior of the constraint set and becomes the global optimum. This case occurs when $\gamma_{\mathrm{L}}h(\zeta^\dagger) \leq \tau_{\mathrm{H}}^\dagger \leq \frac{\gamma_{\mathrm{U}}}{g(\zeta^\dagger)}$.

From Equations (16a) and (16b), the global optimum should satisfy both $3\alpha(\tau_{\mathrm{H}}^\dagger\zeta^\dagger)^2 - \beta(1+4\zeta^{\dagger 2})/(2\tau_{\mathrm{H}}^\dagger\zeta^\dagger)^2 = 0$ and $2\alpha\tau_{\mathrm{H}}^{\dagger 3}\zeta^\dagger - \frac{\beta}{2}\frac{1}{\tau_{\mathrm{H}}^\dagger}\frac{1}{\zeta^{\dagger 3}} = 0$, which leads to $(\tau_{\mathrm{H}}^\dagger\zeta^\dagger)^4 = \beta/4\alpha$.

Therefore, the global optimum $(\zeta^\dagger, \tau_{\mathrm{H}}^\dagger)$ is

$$\tau_{\mathrm{H}}^\dagger = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{4}}; \quad \zeta^\dagger = \frac{1}{\sqrt{2}}. \tag{17}$$

It should be noted that the optimal damping coefficient $\zeta^\dagger$ is independent of the process dynamics and weighting factor.

Case B ($\sigma_1 = \varpi_2 = 0$). The global optimum is located on the constraint $\tau_{\mathrm{H}} = \gamma_{\mathrm{L}}h(\zeta)$ and denoted by $(\zeta^*, \tau_{\mathrm{H}}^*)$, which occurs when $\tau_{\mathrm{H}}^\dagger \leq \gamma_{\mathrm{L}}h(\zeta^\dagger)$ and $\zeta^* \leq \zeta^\nu$.

From Equation (16c), the value of τ_{H} at the global optimum is

$$\tau_{\mathrm{H}}^* = \gamma_{\mathrm{L}}h(\zeta^*) \tag{18}$$

Therefore, from Equation (16b), the value of ζ at the global optimum is expressed as

$$\zeta^* = \frac{1}{\sqrt{2}}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{4}}\frac{1}{\gamma_{\mathrm{L}}h(\zeta^*)} = \frac{1}{\sqrt{2}}\frac{\tau_{\mathrm{H}}^\dagger}{\gamma_{\mathrm{L}}h(\zeta^*)} \tag{19}$$

and can be calculated using a simple root-finding method.

The level controller is usually designed to be $\zeta \geq 0.5$ for avoiding a severe oscillatory response. In this case, one can easily calculate the global optimum $(\zeta^*, \tau_{\mathrm{H}}^*)$ without iteration because $h(\zeta) = 1$. It is clear from Equation (19) that ζ^* is always less than $\zeta^\dagger$ (or $1/\sqrt{2}$) when $\tau_{\mathrm{H}}^\dagger \leq \gamma_{\mathrm{L}}h(\zeta^*)$. This indicates that the contours of the objective function are left skewed in the $\zeta - \tau_{\mathrm{H}}$ plane, as seen from Figure 3.

Case C ($\sigma_2 = \varpi_1 = 0$). The global optimum is located on the constraint $\tau_{\mathrm{H}} = \frac{\gamma_{\mathrm{U}}}{g(\zeta)}$ and denoted by $(\zeta^{**}, \tau_{\mathrm{H}}^{**})$. This case occurs when

$$\tau_{\mathrm{H}}^\dagger \geq \frac{\gamma_{\mathrm{U}}}{g(\xi^\dagger)} \quad \text{and} \quad \zeta^{**} \leq \zeta^\nu.$$

From Equation (16d)

$$\tau_{\mathrm{H}}^{**} = \frac{\gamma_{\mathrm{U}}}{g(\zeta^{**})} \tag{20}$$

Substituting the above equation into Equation (16a) yields

$$3\alpha\tau_{\mathrm{H}}^{**2}\zeta^{**2} - \beta\frac{1}{\tau_{\mathrm{H}}^{**2}}\left(1 + \frac{1}{4\zeta^{**2}}\right) - \frac{\varpi_2\gamma_{\mathrm{U}}}{\tau_{\mathrm{H}}^{**2}} = 0$$

Therefore,

$$\varpi_2 = \frac{3\alpha}{\gamma_U}\tau_H^{**4}\zeta^{**2} - \frac{\beta}{\gamma_U}\left(1+\frac{1}{4\zeta^{**2}}\right)$$

Substituting Equation (20) and ϖ_2 into Equation (16b) gives

$$2\alpha\gamma_U^3 g(\zeta^{**})\zeta^{**} - \frac{\beta}{2\gamma_U}\frac{g(\zeta^{**})^5}{\zeta^{**3}} - \left\{3\alpha\gamma_U^3\zeta^{**2} - \frac{\beta}{\gamma_U}g(\zeta^{**})^4\left(1+\frac{1}{4\zeta^{**2}}\right)\right\}g'(\zeta^{**}) = 0 \quad (21)$$

where $g'(\zeta)$ denotes a derivative of $g(\zeta)$ and is given in Equation (A17) in the Appendix.

Finally, ζ^{**} can be calculated by a simple root-finding method from Equation (21), and then τ_H^{**} can be accordingly found from Equation (20).

Case D ($\sigma_1 = \sigma_2 = 0$). The global optimum denoted by (ζ^ν, τ_H^ν) is located on the vertex point formed by $\tau_H = \gamma_U/g(\zeta)$ and $\tau_H = \gamma_L h(\zeta)$, which occurs either when $\tau_H^\dagger \le \gamma_L h(\zeta^\dagger)$ and $\zeta^* \ge \zeta^\nu$ or when $\tau_H^\dagger \ge \gamma_U/g(\zeta^\dagger)$ and $\zeta^{**} \ge \zeta^\nu$.

The value of ζ at the global optimum point can be found by a simple root-finding method from

$$g(\zeta^\nu)h(\zeta^\nu) = \frac{\gamma_U}{\gamma_L} \quad (22)$$

The value of τ_H at the global optimum point is

$$\tau_H^\nu = \gamma_L h(\zeta^\nu) \quad (23)$$

In summary, when $\gamma_L \ge \gamma_U$, the procedure for finding the global optimum is as follows:

(i) Calculate the unconstrained extreme point $(\zeta^\dagger, \tau_H^\dagger)$. If $\gamma_L h(\zeta^\dagger) \le \tau_H^\dagger \le \gamma_U/g(\zeta^\dagger)$, then $(\zeta^\dagger, \tau_H^\dagger)$ is the global optimum.
(ii) If $\tau_H^\dagger < \gamma_L h(\zeta^\dagger)$, then calculate (ζ^*, τ_H^*) and (ζ^ν, τ_H^ν). If $\zeta^* \le \zeta^\nu$, then (ζ^*, τ_H^*) is the global optimum; if $\zeta^* \ge \zeta^\nu$, then (ζ^ν, τ_H^ν) is the global optimum.
(iii) If $\tau_H^\dagger > \gamma_U/g(\zeta^\dagger)$, then calculate $(\zeta^{**}, \tau_H^{**})$ and (ζ^ν, τ_H^ν). If $\zeta^{**} \le \zeta^\nu$, then $(\zeta^{**}, \tau_H^{**})$ is the global optimum; if $\zeta^{**} \ge \zeta^\nu$, then (ζ^ν, τ_H^ν) is the global optimum.

When a Feasible Region is Unbounded ($\gamma_L < \gamma_U$)

This situation is likely to happen when the $Q'_{o\,\max}$ and/or $H_{\max}$ specifications are mild. Since the feasible region is unbounded, no vertex point is formed by the two constraints for $\zeta > 0.4040$, and accordingly case D does not exist. Also, the inequality conditions $\zeta^* \le \zeta^\nu$ and $\zeta^{**} \le \zeta^\nu$ do not need to be evaluated for cases B and C, respectively.

The procedure for finding a global optimum is as follows:

(i) Calculate the extreme point $(\zeta^\dagger, \tau_H^\dagger)$. If $\gamma_L h(\zeta^\dagger) \le \tau_H^\dagger \le \gamma_U/g(\zeta^\dagger)$, then $(\zeta^\dagger, \tau_H^\dagger)$ is the global optimum.
(ii) If $\tau_H^\dagger < \gamma_L h(\zeta^\dagger)$, then (ζ^*, τ_H^*) is the global optimum.
(iii) If $\tau_H^\dagger > \gamma_U/g(\zeta^\dagger)$, then $(\zeta^{**}, \tau_H^{**})$ is the global optimum.

Optimal PI Parameter Calculation

Once the global optimum is obtained in terms of ζ and τ_H, the corresponding optimal PI parameters can be directly calculated using Equations (7)–(9) as

$$K_c = \frac{(\Delta H)A}{Q_{o\max}\tau_H} \tag{24}$$

$$\tau_I = 4\zeta^2\tau_H \tag{25}$$

Illustrative Examples

The liquid level of a drum with a cross-sectional area of $1\,m^2$ and a working volume $A\Delta H$ of $2\,m^3$ is controlled by a PI controller. The maximum outlet flow rate $Q_{o\max}$ is $4\,m^3$/min. The initial steady-state level is 50% and the nominal flow rates of the inlet and outlet are both $1\,m^3$/min. The expected maximum change in the inlet flow rate ΔQ_i is $1\,m^3$/min. The weighting factor for optimal control is selected as $\omega = 0.8$.

Now, consider the following examples illustrating several representative cases.

Example 1 ($\mathbf{Q'_{o\max}}$ is 3.0 $\mathbf{m^3/min^2}$ and $\mathbf{H_{max}}$ is 1.0 m). Since $H_{\max}Q'_{o\max} > 0.5206(\Delta Q_i^2/A)$, the $(Q'_{o\max}, H_{\max})$ specification given is feasible. γ_L and γ_U are calculated as 0.333 min and 1.0 min, respectively. Therefore, this situation corresponds to the case of $\gamma_L < \gamma_U$, i.e., no vertex point. The values of α and β are 0.4 and 0.01111, respectively. The extreme point can be calculated as $(\zeta^\dagger, \tau_H^\dagger) = (1/\sqrt{2},\ 0.40825)$. From $\gamma_U/g(\zeta^\dagger) = 1.5509$, it satisfies $\gamma_L h(\zeta^\dagger) \le \tau_H^\dagger \le \gamma_U/g(\zeta^\dagger)$. Thus, $(\zeta^\dagger, \tau_H^\dagger)$ is the global optimum. This example belongs to case A, which usually occurs when both two specifications are not so tight. From Equations (24) and (25), the optimal PI parameters are obtained as $K_c = 1.2247$ and $\tau_I = 0.8165$.

Example 2 ($\mathbf{Q'_{o\max}}$ is 1.2 $\mathbf{m^3/min^2}$ and $\mathbf{H_{max}}$ is 1.0 m). In this example, a tighter $Q'_{o\max}$ specification is applied than in Example 1. The specification set is feasible because $H_{\max}Q'_{o\max} > 0.5206(\Delta Q_i^2/A)$. As $\gamma_L < \gamma_U$, the feasible region is unbounded. The extreme point is calculated as $(\zeta^\dagger, \tau_H^\dagger) = (1/\sqrt{2}, 0.6455)$. Since $\gamma_L h(\zeta^\dagger) > \tau_H^\dagger$, this example belongs to case B and the global optimum is on the constraint $\tau_H = \gamma_L h(\zeta)$. The global optimum is obtained as $(\zeta^*, \tau_H^*) = (0.8333, 0.54772)$. The optimal PI parameters are found to be $K_c = 0.6$ and $\tau_I = 1.0$.

Example 3 ($\mathbf{Q'_{o\max}}$ is 4 $\mathbf{m^3/min^2}$ and $\mathbf{H_{max}}$ is 0.2 m). This example has a tight $H_{\max}$ specification with a mild $Q'_{o\max}$ specification. The parameters γ_L, γ_U, α, and β are obtained as 0.25, 0.2, 0.4, and 0.00625, respectively. Since $\gamma_L > \gamma_U$, the vertex point exists and is calculated from Equations (22) and (23) as $(\zeta^\nu, \tau_H^\nu) = (1.3257, 0.25)$. The extreme point is $(\zeta^\dagger, \tau_H^\dagger) = (1/\sqrt{2}, 0.35355)$, whereas the optimum point on $\tau_H = \gamma_U/g(\zeta)$ is $(\zeta^{**}, \tau_H^{**}) = (0.7873, 0.29668)$. Furthermore, $\gamma_U/g(\zeta^\dagger) = 0.3102$. Since it satisfies $\tau_H^\dagger > \gamma_U/g(\zeta^\dagger)$ and $\zeta^{**} \le \zeta^\nu$, this example belongs to case C, and thus $(\zeta^{**}, \tau_H^{**})$ is the global optimum. The optimal PI parameters are obtained as $K_c = 1.6853$ and $\tau_I = 0.7356$.

Example 4 ($\mathbf{Q'_{o\max}}$ is 1.5 $\mathbf{m^3/min^2}$ and $\mathbf{H_{max}}$ is 0.4 m). This is a typical example with tight $H_{\max}$ and $Q'_{o\max}$ specifications. In this case, the global optimum is likely to be on the vertex point. The parameters γ_L, γ_U, α, and β are calculated as

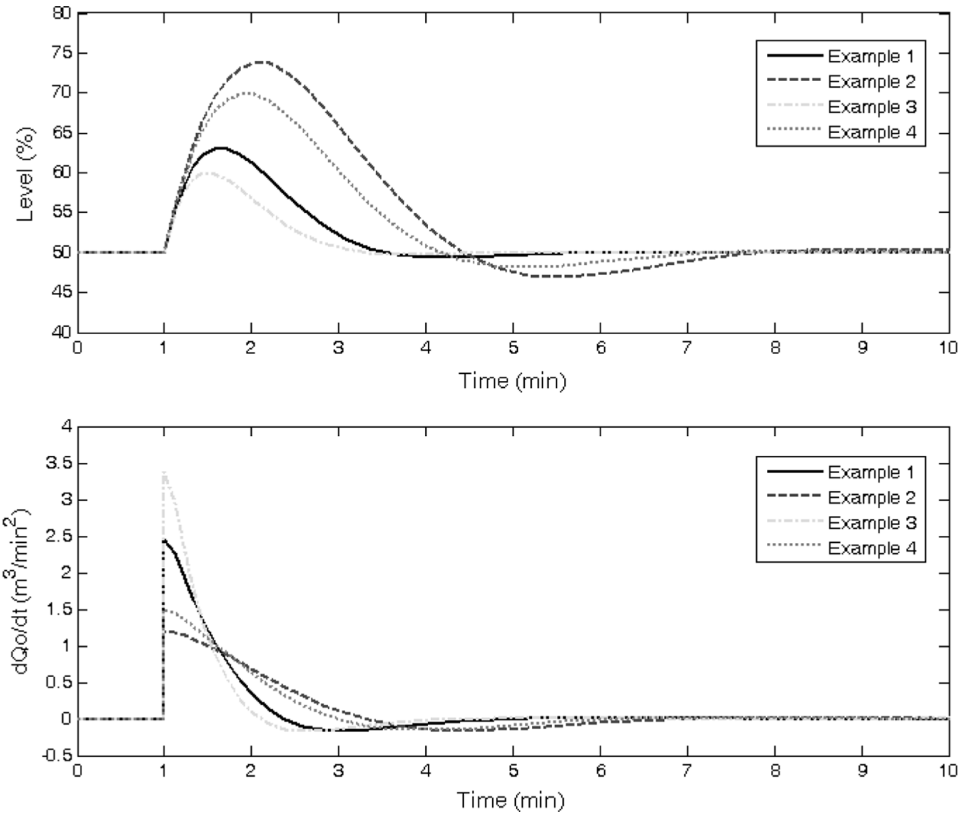

Figure 4. Responses of level and rate of change of outlet flow for examples 1, 2, 3, and 4.

0.66667, 0.4, 0.4, and 0.06667, respectively. Since $\gamma_L > \gamma_U$, the vertex point exists. $(\zeta^\dagger, \tau_H^\dagger)$, (ζ^ν, τ_H^ν), and (ζ^*, τ_H^*) are found to be $(1/\sqrt{2}, 0.57735)$, (0.6029, 0.66667), and (0.61237, 0.66667), respectively. Since $\gamma_L h(\zeta^\dagger) > \tau_H^\dagger$ and $\zeta^* > \zeta^\nu$, the vertex point (ζ^ν, τ_H^ν) is the global optimum. Accordingly, the optimal PI parameters are obtained as $K_c = 0.75$ and $\tau_I = 0.9693$.

Figure 4 compares the responses of the liquid level and the rate of change of the outlet flow for each example. In the simulation, a step change of 1 m^3/min is made in the inlet flow at t = 1. As seen from the figure, the liquid level loop is operated within the constraints by the $H_{\max}$ and $Q'_{o\max}$ specifications in any case. This constraint handling capacity is of great importance in the level control loop because only the minimum inventory is usually provided in a process to achieve the desired dynamic operation (Marlin, 1995).

Comparison with Other Existing Methods

To demonstrate the advantage of the proposed method, the closed-loop performance of the proposed PI controller is also compared with those by Marlin (1995) and the IMC-PI tuning method (Rivera et al., 1986) for example 4. For a fair comparison, each method was adjusted to yield the same maximum level deviation. The PI parameters were found as $K_c = 0.920$ and $\tau_I = 2.1739$ for the Marlin method and $K_c = 1.0$

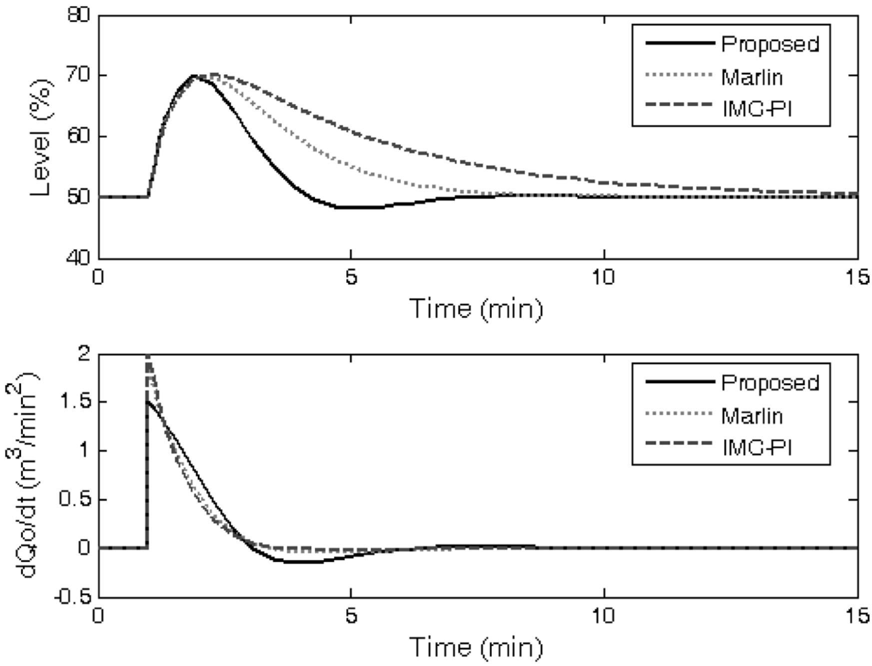

Figure 5. Comparison of responses by the proposed method and the other existing methods for example 4.

and $\tau_I = 4.0$ for the IMC-PI tuning method. As shown in Figure 5, the PI controller by the proposed method gives the smallest maximum rate of change of outlet flow as well as the fastest settling time in the level response. Note that the existing methods compared do not satisfy the two constraints simultaneously.

It is also important in stable and efficient level operation to minimize both the rate of change of the outlet flow and the deviation of the level. To evaluate the optimal control performance, the performance measure in Equation (10a) was also evaluated for each method. As is expected, the proposed PI controller gave the smallest value of 0.169, while the Marlin and IMC-PI methods result in the larger values of 0.181 and 0.215, respectively. All these results clearly show the advantage of the proposed method in the liquid level control.

Effect of Weighting Factor

In the optimal control, the weighting factor w is an important parameter to adjust its performance and robustness. As ω increases, the weight of the controlled variable on the performance measure increases, while the weight of the manipulated variable decreases. Figure 6 compares the responses for the liquid level and the rate of change of the outlet flow rate for various weighting factor settings when $Q'_{o\max}$ and $H_{\max}$ are $1.2\,\mathrm{m}^3/\mathrm{min}^2$ and $1.0\,\mathrm{m}$, respectively. Figure 7 shows how the global optimum location and the extreme point location vary with the weighting factor.

When a weighting factor is set to a small value such as $w = 0.1$, the performance measure of the optimal control is mainly determined by the rate of change in the manipulated flow, i.e., Q'_o. Accordingly, the optimal controller gives a tight response

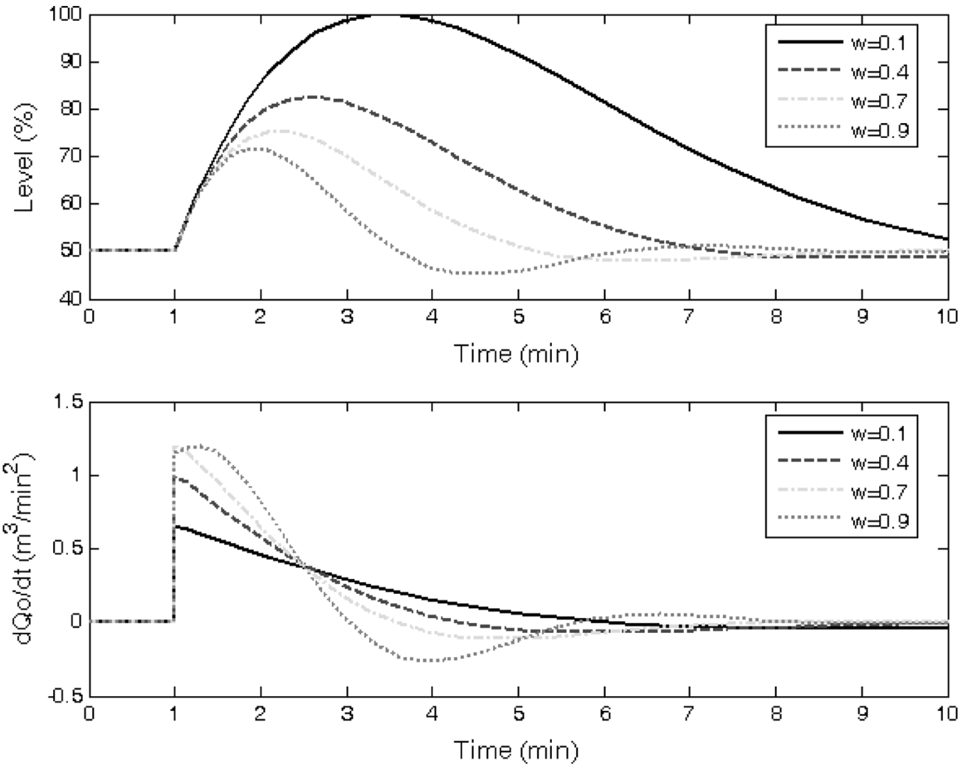

Figure 6. Effect of the weighting factor on the responses of level and rate of change of the outlet flow.

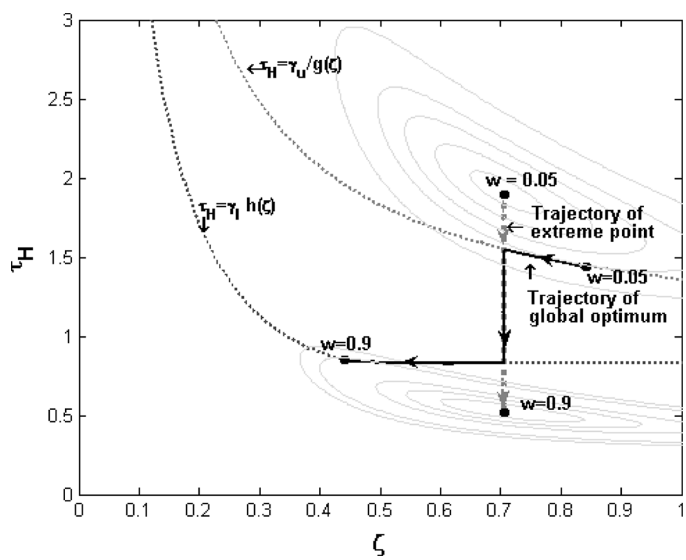

Figure 7. Effect of the weighting factor on the trajectories of extreme and global optimum point.

of Q_o' and the level response is constrained by the $H_{\max}$ specification, as seen from Figure 6. Furthermore, since the extreme point is likely to located above the boundary of $\tau_H = \gamma_U/g(\zeta)$, the global optimal point will accordingly be on $\tau_H = \gamma_U/g(\zeta)$, i.e., case C, as indicated in Figure 7.

As a larger ω is applied, the extreme point shifts to the interior of the two boundaries along the path of $\zeta = 0.7071$, and thus the optimal point comes to be located within the two boundaries, i.e., case A. When w is further increased, the performance measure of optimal control is mainly determined by the controlled variable, i.e., the level response. Therefore, the optimal controller yields a tight level response, and the response is constrained by the $Q'_{o\max}$ specification, as shown in Figure 6. Accordingly, the optimal solution is likely to occur on the constraint $\tau_H = \gamma_L h(\zeta)$ formed by the $Q'_{o\max}$ specification, and the level system belongs to case B. Note that in Figure 6, the proposed PI controller strictly satisfies the constraints by the $H_{\max}$ and $Q'_{o\max}$ specifications regardless of the weighting factor setting.

Conclusions

A constrained optimal control problem for a liquid level system is formulated. The original constrained optimization problem is converted into a simple constrained problem with two independent variables. To obtain the analytical solution for the optimal PI parameters, the constrained optimization problem is further converted into the equivalent unconstrained problem using the classical Lagrangian multiplier method, and the four possible cases associated with the location of the global optimum are analyzed. It is observed that the tightest control specification and the unconstrained optimum response occur at particular values of damping factor, i.e., 0.4040 and 0.7071, respectively. It is shown that the proposed method enables a conventional PI controller to cope with all classes of level control (from tight level control to averaging level control) in a unified manner. The proposed method explicitly deals with the two important control specifications as well as minimizes the optimal performance measure.

Acknowledgment

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Nomenclature

A	cross-sectional area of tank, m^2
H	level deviation, m
ΔH	level transmitter span, m
$H_{\max}$	maximum allowable level deviation, m
K_c	proportional gain, dimensionless
K_L	proportional gain, m^2/min
Q_i	inlet flow rate, m^3/min
ΔQ_i	expected maximum step change in the inlet flow rate, m^3/min
Q_o	outlet flow rate, m^3/min
$Q_{o\max}$	maximum outlet flow rate through a fully open valve, m^3/min
Q_o'	rate of change of outlet flow rate, i.e., $dQ_o(t)/dt$, m^3/min^2

$Q'_{o\max}$	maximum allowable rate of change of outlet flow rate, m^3/min^2
t	time, min

Greek Letters

ζ	damping factor
τ_H	A/K_L, min
τ_I	integral time constant, min
τ_V	holdup time of tank, $(\Delta H)A/Q_{o\max}$, min
Φ	objective function or performance measure for optimal control
ω	weighting factor, $0 < \omega < 1$
ϖ	Lagrangian multiplier

Superscripts

t	tangent point
$\dagger$	extreme point of the objective function
* (or **)	optimum point on the constraint, $\tau_H \geq \gamma_L h(\zeta)$ (or $\tau_H \leq \gamma_U/g(\zeta)$)
ν	optimum point on the vertex

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Appendix: Derivation of the Objective Function and Constraints Given in Equations (11a)–(11c)

Derivation of the Objective Function Φ in Equation (11a)

From Equation (5) with $Q_i(s) = \Delta Q_i/s$ and $H^{set}(s) = 0$, the liquid level response is obtained as

$$H(t) = \frac{\Delta Q_i}{A}\left(\frac{e^{r_1 t} - e^{r_2 t}}{r_1 - r_2}\right) \quad \text{for } r_1 \neq r_2 \tag{A1}$$

where r_1 and r_2 are the roots of the characteristic equation $\tau_H\tau_I s^2 + \tau_I s + 1 = 0$ and

$$\begin{aligned} r_1 &= \frac{-1+\frac{\sqrt{1-\zeta^2}}{\zeta}i}{2\tau_H} \quad r_2 = \frac{-1-\frac{\sqrt{1-\zeta^2}}{\zeta}i}{2\tau_H} \quad \text{for} \quad 0<\zeta<1 \\ r_1 &= \frac{-1+\frac{\sqrt{\zeta^2-1}}{\zeta}}{2\tau_H} \quad r_2 = \frac{-1-\frac{\sqrt{\zeta^2-1}}{\zeta}}{2\tau_H} \quad \text{for} \quad 1<\zeta \end{aligned} \tag{A2}$$

Thus,

$$r_1 r_2 = \frac{1}{\tau_I\tau_H} \qquad r_1 + r_2 = -\frac{1}{\tau_H} \qquad r_1 - r_2 = \frac{1}{\tau_H}\sqrt{1-\frac{4\tau_H}{\tau_I}} = \frac{1}{\tau_H}\sqrt{\frac{\zeta^2-1}{\zeta^2}} \tag{A3}$$

Therefore, the integral square error of the normalized liquid level becomes

$$\omega\int_0^\infty \left[\frac{H(t)}{\Delta H}\right]^2 dt = \omega\left(\frac{\Delta Q_i}{A\Delta H}\right)^2\left(\frac{1}{r_1-r_2}\right)^2\left[\frac{2}{r_1+r_2}-\frac{1}{2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right] = 2\omega\left(\frac{\Delta Q_i}{A\Delta H}\right)^2 \tau_H^3\zeta^2 \tag{A4}$$

From Equation (6) with $Q_i(s) = \Delta Q_i/s$ and $H^{set}(s) = 0$, the outlet flow rate is

$$Q_o(s) = \left(\frac{\tau_I s + 1}{\tau_H\tau_I s^2 + \tau_I s + 1}\right)\frac{\Delta Q_i}{s} \tag{A5}$$

From the property of the Laplace transform for a derivative, $Q_o'(s) \equiv L\left[\frac{dQ_o(t)}{dt}\right] = sQ_o(s)$. Therefore, the rate of change in the outlet flow rate is

$$Q_o'(t) = \Delta Q_i\left(\frac{r_2^2 e^{r_2 t} - r_1^2 e^{r_1 t}}{r_1 - r_2}\right) \quad \text{for } r_1 \neq r_2 \tag{A6}$$

The integral square error of the normalized rate of change in the outlet flow rate becomes

$$\begin{aligned} (1-\omega)\int_0^\infty \left[\frac{Q_o'(t)}{Q'_{o\max}}\right]^2 dt &= (1-\omega)\left(\frac{\Delta Q_i}{Q'_{o\max}}\right)^2\left(\frac{1}{r_1-r_2}\right)^2\left[-\frac{r_2^3}{2}-\frac{r_1^3}{2}+\frac{2r_1^2r_2^2}{r_1+r_2}\right] \\ &= (1-\omega)\left(\frac{\Delta Q_i}{Q'_{o\max}}\right)^2\frac{1}{2\tau_H}\left(1+\frac{1}{4\zeta^2}\right) \end{aligned} \tag{A7}$$

Q.E.D.

Derivation of the Constraint Defined by Equation (11b)

For $\zeta > 1$, it is clear that the largest value of the rate of change in the outlet flow rate for a step disturbance occurs at $t = 0$. Therefore, from Equation (A6),

$$Q'_{o\,peak} = Q_o'(t)|_{t=0} = \frac{\Delta Q_i}{\tau_H} \tag{A8}$$

For $0 < \zeta < 1$, the peak time t_{peak} for the maximum peak of $Q'_o(t)$ can be found by setting the differentiation of Equation (A6) equal to zero and thus be given as

$$t_{peak} = \frac{6\tau_H \zeta}{\sqrt{1-\zeta^2}} \left[\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta} \right) - \frac{\pi}{3} \right] \tag{A9}$$

The maximum of $Q'_o(t)$ can then be obtained by substituting Equation (A9) into Equation (A6) with some mathematical manipulations:

$$Q'_{o\ peak} = \frac{\Delta Q_i}{\tau_H} \cdot \exp\left(-\frac{3\tan^{-1} x - \pi}{x} \right) \left[\frac{1-x^2}{2x} \sin(3\tan^{-1} x) - \cos(3\tan^{-1} x) \right]$$

where $x = \frac{\sqrt{1-\zeta^2}}{\zeta}$.

Since

$$\sin(\tan^{-1} x) = \frac{x}{\sqrt{1+x^2}}, \quad \cos(\tan^{-1} x) = \frac{1}{\sqrt{1+x^2}},$$

$$\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta), \quad \text{and} \quad \cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$$

the maximum of $Q'_o(t)$ can be finally expressed with the following compact form:

$$Q'_{o\ peak} = \frac{\Delta Q_i}{\tau_H} \cdot \frac{\sqrt{1+x^2}}{2} \exp\left(-\frac{3\tan^{-1} x - \pi}{x} \right)$$

However, as can be seen from Equation (A9), t_{peak} has a negative value for $\zeta \geq 0.5$, and thus the maximum of $Q'_o(t)$ still occurs at $t = 0$ for $\zeta \geq 0.5$.

In conclusion, the constraint defined by Equation (10b) can be expressed as

$$\frac{\Delta Q_i}{\tau_H} h(\zeta) \leq Q'_{o\,\max} \tag{A10}$$

where $h(\zeta)$ is:

$$h(\zeta) = \begin{cases} \dfrac{\sqrt{1+x^2}}{2} \exp\left(-\dfrac{3\tan^{-1} x - \pi}{x} \right) & \text{for } 0 < \zeta < 0.5 \\ 1 & \text{for } \zeta \geq 0.5 \end{cases} \tag{A11}$$

where $x = \frac{\sqrt{1-\zeta^2}}{\zeta}$.
Q.E.D.

Derivation of the Constraint Defined by Equation (11c)

The analytical solution of the responses to a step inlet flow change can be obtained from the inverse Laplace transform of Equation (5) as

$$H(t) = \frac{2\Delta Q_i \tau_H}{A} \frac{\zeta}{\sqrt{1-\zeta^2}} \exp\left(-\frac{t}{2\tau_H} \right) \sin(\phi t) \text{ for } \zeta < 1 \left(\text{where } \phi = \frac{1}{2\tau_H} \frac{\sqrt{1-\zeta^2}}{\zeta} \right)$$

$$= \frac{\Delta Q_i}{A} \exp\left(-\frac{t}{2\tau_H} \right) t \qquad \text{for } \zeta = 1$$

$$= \frac{2\Delta Q_i \tau_H}{A} \frac{\zeta}{\sqrt{\zeta^2-1}} \exp\left(-\frac{t}{2\tau_H} \right) \sinh(\phi t) \text{ for } \zeta > 1 \left(\text{where } \phi = \frac{1}{2\tau_H} \frac{\sqrt{\zeta^2-1}}{\zeta} \right) \tag{A12}$$

The peak time t_{peak} for the maximum peak level can be evaluated from the differentiation of Equation (A12) as

$$\begin{aligned} t_{peak} &= \frac{2\tau_H\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) && \text{for} \quad 0<\zeta<1 \\ &= 2\tau_H && \text{for} \quad \zeta = 1 \\ &= \frac{2\tau_H\zeta}{\sqrt{\zeta^2-1}}\tanh^{-1}\left(\frac{\sqrt{\zeta^2-1}}{\zeta}\right) && \text{for} \quad \zeta>1 \end{aligned} \tag{A13}$$

Substituting Equation (A13) into Equation (A12) gives the maximum peak level as

$$H_{peak} = \frac{\Delta Q_i}{A}\cdot\tau_H\cdot g(\zeta) \tag{A14}$$

where $g(\zeta)$ is:

$$\begin{aligned} g(\zeta) &= \frac{2}{\sqrt{1+x^2}}\exp\left(-\frac{\tan^{-1}x}{x}\right) && \text{for} \quad 0<\zeta<1 \quad \left(\text{where } x = \frac{\sqrt{1-\zeta^2}}{\zeta}\right) \\ &= 2\exp(-1) && \text{for} \quad \zeta = 1 \\ &= \frac{2}{\sqrt{1-x^2}}\exp\left(-\frac{\tanh^{-1}x}{x}\right) && \text{for} \quad \zeta>1 \quad \left(\text{where } x = \frac{\sqrt{\zeta^2-1}}{\zeta}\right) \end{aligned} \tag{A15}$$

where $\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$.

Finally, the constraint defined by Equation (10c) can be expressed

$$\frac{\Delta Q_i}{A}\cdot\tau_H\cdot g(\zeta) \le H_{\max} \tag{A16}$$

In addition, $g'(\zeta)$ which denotes $dg(\zeta)/d\zeta$ is given by

$$\begin{aligned} g'(\zeta) &= 2\left(\frac{x^2+1}{x^3}\right)(x-\tan^{-1}x)\exp\left(-\frac{\tan^{-1}x}{x}\right) \qquad \text{for } 0<\zeta<1 \\ &\qquad \left(\text{where } x = \frac{\sqrt{1-\zeta^2}}{\zeta}\right) \\ &= 2\left(\frac{x^2-1}{x^3}\right)(x-\tanh^{-1}x)\exp\left(-\frac{\tanh^{-1}x}{x}\right) \qquad \text{for } \zeta>1 \\ &\qquad \left(\text{where } x = \frac{\sqrt{\zeta^2-1}}{\zeta}\right) \qquad \text{Q.E.D.} \end{aligned} \tag{A17}$$