Implement a constrained optimal control in a conventional level controller—Part 2

Novel tuning method enables a conventional PI controller to explicitly handle the three important operational constraints of a liquid level loop in an optimal manner as well as copes with a broad range of level control from tight to averaging control.

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**Part 1 of this article** outlined the methodology. This part will discuss PI controller design for tightest constraint control and provide some examples.

**PI controller design for tightest constraint control.** For a level loop being operated near the constraint, it is necessary to control the loop on the tightest possible constraint, the so-called "tightest constraint control":

- For a given $H_{\text{max}}$ (or $Q'_{\text{max}}$), the PI controller for the tightest constraint control with respect to $Q'_{\text{max}}$ (or $H_{\text{max}}$) can be designed using either
  \[ \tau_H = \frac{\gamma}{g(L')} \quad \text{or} \quad \tau_H = \gamma_h h(L') \]

  where $L' = 0.404$. In this case, the tightest available $Q_{\text{max}}$ is $1.3606 \Delta Q_i$, or equivalently $L_{\text{min}} \leq 0.404$.

- For a given $(Q'_{\text{max}}, H_{\text{max}})$ set with $\gamma_i \geq \gamma_L$, the PI controller for the tightest constraint control of $Q'_{\text{max}}$ can be obtained by designing it on $L'^i, \tau'^H$. The tightest available $Q_{\text{max}}$ is calculated by $Q_{\text{max}} = \Delta Q_i f(L'^i)$ from Eq. 11 and the tightest available $L_{\text{min}}$ becomes $L_{\text{min}}^i = L'^i$.

  - For a given $(Q'_{\text{max}}, H_{\text{max}})$ set, the PI controller that gives the tightest $Q'_{\text{max}}$ can be designed based on $(L_{\text{min}}^i, \tau'^H)$. The tightest available $Q'_{\text{max}}$ is calculated by
    \[ Q'_{\text{max}} = \frac{\Delta Q_i f(L'^i)}{\gamma_{\max}} \]

  - For a given $(Q'_{\text{max}}, Q_{\text{max}})$ set, the PI controller that gives the tightest $H_{\text{max}}$ can be obtained using $(L_{\text{min}}^i, \tau'^H)$. The tightest available $H_{\text{max}}$ is obtained as
  \[ H_{\text{max}} = \frac{\Delta Q_i f(L'^i)}{\gamma_{\max}} \]

**Illustrative examples.** Consider a liquid level loop: $A = 1$ $\text{m}^2$, $\Delta H_{\text{max}} = 2$ m, $Q_{\text{max}} = 4$ $\text{m}^3/\text{min}$, $Q_{\text{max}} = 5$ $\text{m}^3/\text{min}$. The initial steady-state level is assumed to be at 50%. The expected maximum $\Delta Q_i$ is $1$ $\text{m}^3/\text{min}$, and $w$ is chosen as 0.8.

Seven examples are investigated to illustrate the corresponding seven possible cases. The constraint sets and the resulting optimal PI parameters are listed in Table 2. Figs. 5a and 5b show the responses afforded by the resulting optimal PI controller. In the simulation, a step change in the inflow is made at $t = 1$. As seen
from the figures, the level loop is optimally controlled and strictly satisfies the given three constraints in every case.

It should be noted that the PI parameters for examples 4, 6 and 7 also imply those for the tightest constraint control with respect to \( Q_{\text{max}}^o \), \( Q_{\text{max}}^o \), and \( H_{\text{max}} \) when the \((Q_{\text{max}}^o , H_{\text{max}})\), \((Q_{\text{max}}^o , H_{\text{max}})\) and \((Q_{\text{max}}^o , H_{\text{max}})\) set are assumed to be given as those of examples 4, 6 and 7 in Table 2, respectively. The tightest \( Q_{\text{max}}^o \), \( Q_{\text{max}}^o \), and \( H_{\text{max}} \) specifications are calculated as 1.2476 m^3/min, 4.7746 mV min^-1 and 0.2387 m, respectively. The responses of examples 4, 6 and 7 in Table 2, respectively. The tightest \( Q_{\text{max}}^o \), \( Q_{\text{max}}^o \), and \( H_{\text{max}} \) imply those for the tightest constraint control with respect to \( Q_{\text{max}}^o \), \( Q_{\text{max}}^o \), and \( H_{\text{max}} \) when the \((Q_{\text{max}}^o, H_{\text{max}})\), \((Q_{\text{max}}^o, H_{\text{max}})\) and \((Q_{\text{max}}^o, H_{\text{max}})\) set are assumed to be given as those of examples 4, 6 and 7 in Table 2, respectively. The tightest \( Q_{\text{max}}^o \), \( Q_{\text{max}}^o \), and \( H_{\text{max}} \) specifications are calculated as 1.2476 m^3/min, 4.7746 mV min^-1 and 0.2387 m, respectively. The responses of examples 4, 6 and 7 in Figs. 5a and 5b clearly meet these tightest constraints.

**End of series:** Part 1, January 2010.

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**NOMENCLATURE**

<table>
<thead>
<tr>
<th>Roman letters</th>
<th>Greek letters</th>
<th>Superscripts</th>
</tr>
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<tbody>
<tr>
<td>( A )</td>
<td>( H )</td>
<td>( t )</td>
</tr>
<tr>
<td>Drum cross-sectional area, ( m^2 )</td>
<td>Level deviation from setpoint, m</td>
<td>Tangent point</td>
</tr>
<tr>
<td>( \Delta H_{\text{opt}} )</td>
<td>( \xi )</td>
<td>( \gamma )</td>
</tr>
<tr>
<td>Level transmitter span, m</td>
<td>Damping factor</td>
<td>Optimum point of the objective function</td>
</tr>
<tr>
<td>( H_{\text{max}} )</td>
<td>( \tau )</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>Maximum allowable level deviation from setpoint, m</td>
<td>Integral time constant, min</td>
<td>Slack variable</td>
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<tr>
<td>( K_f )</td>
<td>( \tau_f )</td>
<td></td>
</tr>
<tr>
<td>Proportional gain (dimensional), ( m^3/\text{min} )</td>
<td>Apparent holdup time, ( A/K_f ) or ( \tau_f/K_f ), min</td>
<td>( \gamma_f = f(\xi) )</td>
</tr>
<tr>
<td>( Q )</td>
<td>( \tau_H )</td>
<td></td>
</tr>
<tr>
<td>Inflow change, ( m^3/\text{min} )</td>
<td>( \tau_H = \gamma_f h(\xi) )</td>
<td>( \tau_H = \gamma_f h(\xi) )</td>
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<tr>
<td>( Q_{\text{max}} )</td>
<td>( \gamma_H )</td>
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<td>Outflow change, ( m^3/\text{min} )</td>
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<tr>
<td>( Q_{\text{max}} )</td>
<td></td>
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<tr>
<td>Maximum allowable change in outflow, ( m^3/\text{min} )</td>
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<td></td>
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<td>( Q_{\text{max}} )</td>
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<tr>
<td>Maximum achievable outflow through the outlet valve, ( m^3/\text{min} )</td>
<td>( \alpha )</td>
<td></td>
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<tr>
<td>( Q_{\text{max}} )</td>
<td></td>
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| \( \Delta Q \) | | | | Dennis, J.B. (1959), Mathematical Programs and Electron Networks, John Wiley & Sons, New York.

**TABLE 2.** Constraint specifications and PI parameters for the illustrative examples

<table>
<thead>
<tr>
<th>Example</th>
<th>Case</th>
<th>Specification</th>
<th>PI parameter</th>
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</thead>
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<tr>
<td>1</td>
<td>A</td>
<td>( Q_{\text{max}} )</td>
<td>4.0, 1.0, 1.3</td>
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<tr>
<td>2</td>
<td>B</td>
<td>( H_{\text{max}} )</td>
<td>2.3, 1.0, 1.3</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
<td>( Q_{\text{max}} )</td>
<td>5.0, 0.15, 1.3</td>
</tr>
<tr>
<td>4</td>
<td>D</td>
<td>( Q_{\text{max}} )</td>
<td>4.0, 0.15, 1.3</td>
</tr>
<tr>
<td>5</td>
<td>E</td>
<td>( Q_{\text{max}} )</td>
<td>5.0, 0.3, 1.15</td>
</tr>
<tr>
<td>6</td>
<td>F</td>
<td>( Q_{\text{max}} )</td>
<td>6.0, 0.15, 1.15</td>
</tr>
<tr>
<td>7</td>
<td>G</td>
<td>( Q_{\text{max}} )</td>
<td>3.0, 0.5, 1.15</td>
</tr>
</tbody>
</table>

\( V_f \) Drum surge volume, \( DA H_{\text{max}} \), \( m^3 \)

\( \xi \) Damping factor

\( \tau \) Integral time constant, min

\( \tau_p \) Apparent holdup time, \( A/K_f \) or \( \tau_f/K_f \), min

\( \omega \) Weighting factor of objective function, \( 0 < \omega < 1 \)

\( \phi \) Objective function or performance measure for optimal control

\( \alpha \) Lagrangian multiplier

\( \delta \) Slack variable
Appendix. Derivation of performance measure and inequality constraints given in Eqs. 11a–d

**Derivation of the performance measure** \( \phi \) given by Eq. 11a

For a step change of magnitude \( \Delta Q_i \) in the inflow, \( H(t) \) and \( Q'_o(t) \) are obtained from Eqs. 4 and 5, respectively, as follows:

\[
H(t) = \frac{\Delta Q_i}{A} \left( e^{r_1 t} - e^{r_2 t} \right) \quad \text{for } r_1 \neq r_2 \tag{A1}
\]

\[
Q'_o(t) = \Delta Q_i \left( \frac{2 e^{r_1 t} - e^{r_2 t}}{r_1 - r_2} \right) \quad \text{for } r_1 \neq r_2 \tag{A2}
\]

where \( r_1 \) and \( r_2 \) are the roots of the characteristic equation \( \tau_H r^2 + \tau_H + 1 = 0 \).

Therefore, the performance measure for optimal control is:

\[
\omega \int_0^\infty \frac{H(t)^2}{\Delta H_{\text{up}}^2} \, dt + (1 - \omega) \int_0^\infty \frac{Q'_o(t)^2}{Q'_{o \text{ max}}^2} \, dt
\]

\[
= \omega \left( \frac{\Delta Q_i}{\Delta H_{\text{up}}} \right)^2 \left[ \frac{1}{r_1 - r_2} + \frac{1}{r_1 + r_2} \right] + (1 - \omega) \left( \frac{\Delta Q_i}{Q'_{o \text{ max}}} \right)^2 \left[ \frac{1}{r_1 - r_2} - \frac{1}{r_1 + r_2} \right]
\]

\[
= 2\omega \frac{\Delta Q_i}{\Delta H_{\text{up}}} \tau_H^3 \zeta^2 + (1 - \omega) \frac{\Delta Q_i}{Q'_{o \text{ max}}} \zeta^2 \times \frac{1}{1 + \frac{2}{\tau_H} \zeta^2} \tag{A3}
\]

**Derivation of the constraint given by Eq. 11d**

The outflow response to a step change in the inflow is obtained from the inverse Laplace transform of Eq. 5 as follows:

\[
Q'_o(t) = \Delta Q_i \left[ 1 + \exp \left( -\frac{t}{2\tau_H} \right) \right] \left( -\cos(\phi t) + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin(\phi t) \right)
\]

for \( \zeta < 1 \) where \( \phi = \frac{1}{2\tau_H} \sqrt{1 - \zeta^2} \)

\[
= \Delta Q_i \left[ -\frac{1}{2\tau_H} + \exp \left( -\frac{t}{2\tau_H} \right) \right] + 1 \quad \text{for } \zeta = 1
\]

\[
= \Delta Q_i \left[ 1 + \exp \left( -\frac{t}{2\tau_H} \right) \right] - \cos(\phi t) \frac{\zeta}{\sqrt{\zeta^2 - 1}} \sinh(\phi t)
\]

for \( \zeta > 1 \) where \( \phi = \frac{1}{2\tau_H} \sqrt{\zeta^2 - 1} \)

\[
Q'_{o \text{ peak}} = \Delta Q_i f(\zeta) \quad \text{(A5)}
\]

where \( f(\zeta) \) is:

\[
f(\zeta) = 1 + \exp \left( -\frac{2}{\tau_H} \right) \frac{1}{\phi} \sin(\phi t)
\]

for \( 0 < \zeta < 1 \) where \( \phi = \frac{1}{\tau_H} \sqrt{1 - \zeta^2} \)

\[
= 1 + \exp(-2) \quad \text{for } \zeta = 1
\]

\[
= 1 + \exp \left( \frac{2\tanh^{-1} x}{x} \right)
\]

for \( \zeta > 1 \) where \( x = \frac{\sqrt{\zeta^2 - 1}}{\zeta} \) (A6)

Therefore, the constraint in Eq. 11d can be expressed as:

\[
\Delta Q_i f(\zeta) \leq Q'_{o \text{ max}} \tag{A7}
\]

**Derivation of the constraint given by Eq. 11b**

The rate of change of the outflow, \( Q'_o(t) \), is obtained by differentiating Eq. 4. The peak of \( Q'_o(t) \) can be found from \( dQ'_o(t)/dt \) in terms of \( \zeta \) and \( \tau_H \):

\[
Q'_{o \text{ peak}} = \frac{\Delta Q_i}{\tau_H} h(\zeta) \tag{A8}
\]

where \( h(\zeta) \) is:

\[
h(\zeta) = \frac{1 + x^2}{2} \exp \left( -\frac{3\tan^{-1} x - \pi}{x} \right)
\]

for \( 0 < \zeta < 0.5 \) where \( x = \frac{1}{\sqrt{1 - \zeta^2}} \)

\[
= 1 \quad \text{for } \zeta > 0.5 \quad (A9)
\]

Therefore, the constraint given by Eq. 11b can be expressed as:

\[
\frac{\Delta Q_i}{\tau_H} h(\zeta) \leq Q'_{o \text{ max}} \tag{A10}
\]

**Derivation of the constraint given by Eq. 11c**

The analytical solution of the level responses can be obtained from the inverse Laplace transform of Eq. 4 as:

\[
H(t) = \frac{\Delta Q_i}{A} \frac{1}{\phi} \exp \left( -\frac{t}{2\tau_H} \right) \sin(\phi t)
\]

for \( \zeta < 1 \) where \( \phi = \frac{1}{2\tau_H} \sqrt{1 - \zeta^2} \)

\[
= \frac{\Delta Q_i}{A} \frac{1}{\phi} \exp \left( -\frac{t}{2\tau_H} \right) \sinh(\phi t) \quad \text{for } \zeta = 1
\]

\[
= \frac{\Delta Q_i}{A} \frac{1}{\phi} \exp \left( -\frac{t}{2\tau_H} \right) \sin(\phi t)
\]

for \( \zeta > 1 \) where \( \phi = \frac{1}{2\tau_H} \sqrt{\zeta^2 - 1} \)

\[
H'_{o \text{ peak}} = \frac{\Delta Q_i}{A} \tau_H g(\zeta) \tag{A11}
\]

The peak of \( H(t) \) can be found in terms of \( \zeta \) and \( \tau_H \):

\[
H'_{o \text{ peak}} = \frac{\Delta Q_i}{A} \tau_H g(\zeta)
\]

\( \text{(A12)} \)
where $g(\xi)$ is:

$$g(\xi) = \frac{2}{\sqrt{1 + x^2}} \exp\left(-\frac{\tan^{-1} x}{x}\right)$$

for $0 < \xi < 1$ where $x = \frac{\sqrt{1 - \xi^2}}{\xi}$

$$= 2 \exp(-1) \quad \text{for } \xi = 1$$

$$= \frac{2}{\sqrt{1 - x^2}} \exp\left(-\frac{\tanh^{-1} x}{x}\right)$$

for $\xi > 1$ where $x = \frac{\sqrt{x^2 - 1}}{\xi}$ (A13)

Finally, the constraint given by Eq. 11c can be expressed as:

$$\frac{\Delta Q}{A} \cdot \tau_H g(\xi) \leq H_{\text{max}}$$ (A14)

Furthermore, $g'(\xi)$, which denotes $\frac{dg(\xi)}{d\xi}$, is given by:

$$g'(\xi) = 2 \left(\frac{x^2 + 1}{x^3}\right) \left(x - \tan^{-1} x\right) \exp\left(-\frac{\tan^{-1} x}{x}\right)$$

for $0 < \xi < 1$ where $x = \frac{\sqrt{1 - \xi^2}}{\xi}$

$$= 2 \left(\frac{x^2 - 1}{x^3}\right) \left(x - \tanh^{-1} x\right) \exp\left(-\frac{\tanh^{-1} x}{x}\right)$$

for $\xi > 1$ where $x = \frac{\sqrt{x^2 - 1}}{\xi}$ (A15)

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