Analytical Design of Multi-Loop PI Controllers for Interactive Multivariable Processes

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In this paper, a simple and efficient design method of multi-loop PI controllers is proposed by extending the Maclaurin series approach to a multi-loop control system. The controller parameter of a multi-loop system is related to that of a SISO non-interacting system with an interaction factor in the simple multiplication form. Based on this relation, analytical tuning rules for a multi-loop PI controller are derived for several representative process models. In order to improve both performance and robustness of the multi-loop control system, the multi-loop Ms criterion is utilized as a performance cost function. The simulation studies confirm the effectiveness of the proposed method.

Introduction

Multivariable or multi-input/multi-output (MIMO) systems are frequently encountered in the chemical and process industries. Despite the considerable work on advanced multivariable controllers, multi-loop proportional-integral-derivative (PID) controllers are more favored in controlling MIMO systems with modest interactions because of their adequate performance and several practical advantages that include simple implementation, fewer tuning parameters, loop failure tolerance, and ease of structure understanding. In a multi-loop system, once a loop pairing is fixed, control performance is determined mainly by tuning each multiple single-loop PI/PID controller. However, the complex interactive nature of MIMO systems makes the proper tuning of multi-loop PI/PID controllers much more difficult than that of single-loop PI/PID controllers.

For this reason, extensive effort has been focused on how to effectively take into account interactions until some stability criteria are satisfied. The best-known method of this type is the biggest log-modulus tuning (BLT) method suggested by Luyben (1986). Initially, each controller of the multi-loop system is determined via the Ziegler–Nichols (Z–N) rules (1942) for each control loop, ignoring interactions. Then, all settings are detuned by a common factor, $F$, determined via a Nyquist-like plot of the closed-loop characteristic polynomial. This method was later extended by Monica et al. (1988), where instead of adjusting $F$, the individual weighting factors on the basis of the level of imbalance in the controller detuning are introduced for the detuning of the corresponding control loops.

In the independent loop methods, each controller is mainly designed based on the corresponding open-loop and closed-loop transfer functions, while the process interactions are taken into account by satisfying some inequality constraints of stability. Various types of constraints imposed on the independent design have been given by a number of authors (Grosdidier and Morari, 1987; Skogestad and Morari, 1989; Lee et al., 2003; Truong et al., 2007). The independent loop methods are successfully utilized when the system is decoupled in space (the process is diagonally dominant). The main advantage of the independent loop methods is that loop failure tolerance is automatically guaranteed. However, it is conservative due to its assumption, which does not exploit the detailed information about the controller dynamics in other loops.

In the sequential loop tuning methods (Mayne, 1973; Hovd and Skogestad, 1994), each controller is sequentially designed for ignoring its multivariable nature. Initially, a SISO controller is designed for one
pair of input and output variables, and the loop is then
closed. Then, a second SISO controller is designed for
a second pair of variables while the first control loop
remains closed, and so on. The sequential loop tuning
methods can be used for complex interactive problems
where the independent loop method does not work. A
potential disadvantage of this approach is that loop
failure tolerance is not guaranteed when the previous
loops fail. In particular, the individual loops are not
guaranteed to be stable. When the system outputs can
be decoupled in time, sequential loop tuning methods
can be effectively used for the design of multi-loop
controllers.

The relay auto-tuning for the multi-loop control
system has been widely used by a number of authors
(Loh et al., 1993; Shen and Yu, 1994; Halevi et al.,
1997). This approach directly combines the efficiency
of single-loop relay auto-tuning and sequential loop
tuning method, wherein the control loops can be tuned
in a sequential manner or simultaneously. The disad-
vantages of this approach are that unsatisfactory per-
formance occurs when the loops are not properly paired
and loop failure tolerance cannot be easily checked.

Although a number of applicable methods have
been reported, the design and tuning of multi-loop PID
controllers in practice still remains quite difficult and
challenging. Most of them require complex iterative
steps with non-analytical forms (Luyben, 1986;
Skogestad and Morari, 1989; Loh et al., 1993); the in-
teraction effect implicitly embedded into the multi-loop
controller parameters makes it difficult to figure out
how the interaction affects the controller parameters
compared with the non-interacting case.

In this paper, our aim is to develop an analytical
and efficient design method of multi-loop PI control-
ners that can take the interactions fully into account in
a simple manner by expressing the controller param-
eters in terms of those in the non-interacting case. It
was found that the multi-loop controller parameters can
be expressed as a simple multiplication of the inter-
action factor and controller parameters of the non-in-
teracting case. Based on the relations of the multi-loop
controller gains between the interacting and non-in-
teracting case, the analytical tuning rules for multi-
loop PI controllers are derived for several representa-
tive process models. Several multivariable industrial
processes with different interaction characteristics were
employed to demonstrate the effectiveness of the pro-
posed method compared to the existing methods.

1. Multi-Loop PI Controller Design

Consider a multi-loop feedback system as shown
in Figure 1, where $R_i$, $U_i$, and $Y$ denote the set-point,
and manipulated and controlled variables, respectively,$G(s)$ denotes an $n$-input and $n$-output open-loop stable
multivariable process, which is controlled by a multi-
loop (diagonal) controller $\tilde{G}_c(s)$. The sensitivity and
complementary sensitivity function matrices $S(s)$ and
$H(s)$ are given respectively as follows

$$
S(s) = \left[I + G(s)\tilde{G}_c(s)\right]^{-1} \tag{1}
$$

$$
H(s) = I - S(s) = \left[I + G(s)\tilde{G}_c(s)\right]^{-1} G(s)\tilde{G}_c(s) \tag{2}
$$

The process $G(s)$ can be expanded in a Maclaurin se-
ries as given by Eq. (3).

$$
G(s) = G_0 + G_1(s) + O(s^2) \tag{3}
$$

where $G_0 = G(0)$ and $G_1 = G'(0)$.

The multi-loop controller $\tilde{G}_c(s)$ with the integral
term can also be written in a Maclaurin series as given
by Eq. (4).

$$
\tilde{G}_c(s) = \frac{1}{s} \left[\tilde{G}_c0 + \tilde{G}_c1s + O(s^2)\right] \tag{4}
$$

where $\tilde{G}_c0$ and $\tilde{G}_c1$ correspond to the integral and
proportional terms of the multi-loop PI controller, re-
spectively.

Although the controller given by Eq. (4) includes
an infinite number of high-order $s$ terms, the first two
terms ($\tilde{G}_c0$ and $\tilde{G}_c1$) are often sufficient to achieve
the desired closed-loop performance at the low and
middle frequencies that are much more important than
high frequencies in an actual control situation. Conse-
quently, the controller can be approximated with the
first two terms as the standard multi-loop PI control-
er, given by Eq. (5).

$$
\tilde{G}_c(s) = \frac{1}{s} \left[\tilde{K}_1 + \tilde{K}_c\right] \tag{5}
$$

where $\tilde{K}_c = \tilde{G}_c1$ and $\tilde{K}_1 = \tilde{G}_c0$. 

![Figure 1 Multi-loop control system](image-url)
Substituting Eqs. (3) and (4) into Eq. (2), and rearranging it, the complementary sensitivity function matrix (sometimes known as the closed-loop transfer function matrix) in a Maclaurin series is obtained as Eq. (6).

$$H(s) = I - \left( G_a G_{co} \right)^{-1} s + \left( G_a G_{co} \right)^{-1} \left( I + G_a G_{co} + G_a G_{ci} \right) \left( G_a G_{co} \right)^{-1} s^2 + O(s^3)$$

According to the design strategy of the multi-loop IMC-PID controller (Economou and Morari, 1986; Grosdidier and Morari, 1986, 1987; Skogestad and Morari, 1989; Lee et al., 2004), the desired closed loop response $R_i$ of the $i$-th loop is typically given by Eq. (7).

$$R_i = f G_{wi}(s) = \frac{G_{wi}(s)}{\lambda, s + 1}$$

where $f = 1/(\lambda s + 1)$ is a low-pass filter, of which $\lambda$ is the design parameter as well as the desired closed-loop time constant for the tradeoffs between performance and robustness. The exponent $r_i$ is a positive integer selected to be large enough to make the IMC controller proper. $G_{wi}(s)$ is the non-invertible (non-minimum phase) part of $G_{wi}$ that contains any time delays and right-half-plane zeros, and is chosen to be the all pass form for the best least-squares response. For the controlled variable to track its set-point, $G_{wi}(s)$ is specified so that its steady-state gain is 1.

The desired closed-loop response matrix is given by Eq. (8).

$$\tilde{R}(s) = \text{diag}[R_1, R_2, \ldots, R_n]$$

Expanding $\tilde{R}(s)$ in a Maclaurin series gives Eq. (9).

$$\tilde{R}(s) = \tilde{R}_0 + \tilde{R}_1 s + \tilde{R}_2 s^2 + O(s^3)$$

where $\tilde{R}_0 = \tilde{R}(0) = I$, $\tilde{R}_1 = \tilde{R}'(0)$, and $\tilde{R}_2 = \tilde{R}''(0)/2$.

Comparing each diagonal element of $H(s)$ and $\tilde{R}(s)$ in Eqs. (6) and (9) for the first two $s$ terms ($s$ and $s^2$), yields Eqs. (10) and (11).

$$\tilde{R}_1 = \text{diag}\left\{ \left( G_a G_{co} \right)^{-1} \right\}$$

$$\tilde{R}_2 = \text{diag}\left\{ \left( G_a G_{co} \right)^{-1} \left( I + G_a G_{co} + G_a G_{ci} \right) \left( G_a G_{co} \right)^{-1} \right\}$$

Accordingly, the integral term of the multi-loop PI controller can be constituted by rearranging Eq. (10) as Eq. (12).

$$\tilde{K}_i = \tilde{G}_{ci} = \text{diag}(G_{wi})\tilde{R}_i^{-1}$$

If a MIMO system has no process interaction, a MIMO system is identical to a set of independent SISO systems. Thus, the integral term $\tilde{k}_i$ of a multi-loop PI controller for the non-interacting case can be derived as Eq. (13), thus arriving of Eq. (14).

$$\tilde{k}_i = -\left( \tilde{R}, \tilde{G}_o \right)^{-1} = -\tilde{G}_o \tilde{R}_i^{-1}$$

$$\tilde{R}_i^{-1} = -\tilde{G}_o \tilde{R}_i^{-1}$$

Substituting Eq. (14) into Eq. (12), the integral term of the multi-loop controller for the interacting case can be related to the $\tilde{k}_i$ for the non-interacting, as follows shown in Eq. (15).

$$\tilde{K}_i = \text{diag}(G_{wi}) G_i \tilde{k}_i = G_{wi} \frac{G''(0)}{G(0)} \tilde{k}_i$$

where $|G(0)|$ is the determinant of $G(0)$ and the scalar $G''(0)$ is the $i$-th diagonal element of the adjoint of $G(0)$.

The well-known relative gain array (RGA) introduced by Bristol (1966) is calculated as: $\Lambda = \tilde{G}(0) \otimes \tilde{G}(0)^{-1}$. Thus, the RGA with diagonal elements only is expressed as $\Lambda = \tilde{G}_i G_i''(0)/|G(0)|$. Therefore, Eq. (15) can be compactly rewritten as Eq. (16).

$$\tilde{K}_i = \Lambda \tilde{k}_i$$

Note that the definition of relative gain is given by Eq. (17).

$$\Lambda_{ij} = \left( \frac{\partial Y_j}{\partial U_i} \right) = \frac{\text{open-loop gain}}{\text{closed-loop gain}}$$

It is interesting that the integral gains of multi-loop PI controllers between the interacting and non-interacting (SISO) cases are simply related with the RGA. Equation (16) is also consistent with engineering intuition. For example, consider the case where the relative gain, $\Lambda_{ij}$, is greater than 1 (the closed-loop gain of a loop is smaller than the open-loop gain). In this case, a larger control action $U_i$ is required to achieve a given change in $Y_i$ in the closed-loop than in the open-loop. Therefore, a larger controller gain should be used in the interacting case than in the non-interacting case, as Eq. (16) indicates.
It can be seen from Eq. (4) that the impact of $\tilde{G}_{c1}$ in $G_{c1}$ becomes dominant at high frequencies, whereas it is relatively less significant at low frequencies. Thus, it is desirable for $G_{c1}$ to be designed based on the process characteristics at high frequency. Since $|G(j\omega)|G_{c1}(j\omega)| \ll 1$ at high frequencies, the closed-loop transfer function matrix given by Eq. (2) can be approximated to $H(s) = G(s)G_{c}(s)$, which indicates the transmission interaction is mostly eliminated by low-pass filtering through the processes in the other loop (Lee et al., 2004). This means $G_{c1}$ can be reasonably designed by considering only the diagonal elements in $G(s)$. From this feature at high frequencies, Eq. (11) can be approximated for designing the proportional term as Eq. (18).

$$\Phi \equiv \text{diag}\left\{ (G_0G_{cn})^{-1} \left[ I + G_0G_{cn} + G_0G_{c1} \right] \right\} \cdot \text{diag}\left\{ (G_0G_{cn})^{-1} \right\}$$

Equation (18)

Therefore, the proportional term of the multi-loop PI controller can be derived by rearranging Eq. (18) as Eq. (19).

$$\tilde{K}_c = \tilde{G}_{c1} = \tilde{R}_i^{-1}G_0^{-1}\left[ \tilde{R}_i^{-1}\tilde{R}_2 - \tilde{R}_i + \text{diag}(G_0^{-1})\tilde{G}_r \right]$$

Equation (19)

For a non-interacting MIMO scenario, the proportional term $\tilde{k}_c$ of a multi-loop PI controller can be expressed as Eq. (20).

$$\tilde{k}_c = \tilde{R}_i^{-1}G_0^{-1}\left[ \tilde{R}_i^{-1}\tilde{R}_2 - \tilde{R}_i + \tilde{G}_r^{-1}\tilde{G}_1 \right]$$

Equation (20)

By rearranging Eq. (20), the inverse of $\tilde{R}_i$ can be written as Eq. (21).

$$\tilde{R}_i^{-1} = \left[ (\tilde{R}_i^{-1}\tilde{R}_2 - \tilde{R}_i + \tilde{G}_r^{-1}\tilde{G}_1) \right] G_0^{-1}\tilde{k}_c$$

Equation (21)

Substituting Eq. (21) into Eq. (19), the proportional term $\tilde{K}_c$ of a multi-loop PI controller can be linked to $\tilde{k}_c$, as follows in Eq. (22).

$$\tilde{K}_c = \tilde{R}_i^{-1}\tilde{R}_2 - \tilde{R}_i + \text{diag}(G_0^{-1})\tilde{G}_r \cdot \left( \tilde{R}_i^{-1}\tilde{R}_2 - \tilde{R}_i + \tilde{G}_r^{-1}\tilde{G}_1 \right)^{-1} \tilde{k}_c = \tilde{G}_c$$

Equation (22)

where $\tilde{G}_c$ is the closed-loop dynamic interaction matrix, defined as Eq. (23).

$$\tilde{G}_c = \left( \tilde{R}_i^{-1}\tilde{R}_2 - \tilde{R}_i + \text{diag}(G_0^{-1})\tilde{G}_r \right) \cdot \left( \tilde{R}_i^{-1}\tilde{R}_2 - \tilde{R}_i + \tilde{G}_r^{-1}\tilde{G}_1 \right)^{-1}$$

Equation (23)

As seen from Eqs. (16) and (22), the controller gains of a multi-loop PI controller can be simply related to those of a SISO PI controller by introducing the interaction factors such as $\Lambda_i$ for the integral gain and $\Phi_i$ for the proportional gain, respectively. Note that both $\Lambda_i$ and $\Phi_i$ do not require a priori knowledge of other controllers, but include only information of process dynamics.

2. Analytical Multi-Loop PI Controller Tuning Rules

The most widely used dynamic models of industrial processes are first-order-plus-dead-time (FOPDT) and second-order-plus-dead-time (SOPDT). In this section, the analytical tuning rules of the multi-loop PI controller are derived for these models.

First, consider an interactive multivariable process with FOPDT dynamics, represented as the following matrix in Eq. (24).

$$G(s) = \left[ \begin{matrix} g_{11}(s) & g_{12}(s) & \cdots & g_{1n}(s) \\ g_{21}(s) & g_{22}(s) & \cdots & g_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1}(s) & g_{n2}(s) & \cdots & g_{nn}(s) \end{matrix} \right]$$

Equation (24)

where $g_{ij}(s) = K_{ij}e^{-\theta_i/(\tau_j + 1)}$, $i, j = 1, 2, ..., n$, of which $K_{ij}$, $\tau_j$, and $\theta_i$ represent the process gain, time constant, and time delay of the $(i, j)$-th element, respectively.

The process gain matrix is defined as Eq. (25).

$$G_0 = G(0) = \left[ \begin{matrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{matrix} \right]$$

Equation (25)

Therefore, the diagonal matrix of $G_0$ can be obtained as Eq. (26).

$$G_0 = \text{diag}\{K_i\}$$

Equation (26)

The diagonal matrix of $G_1$ can be computed by Eq. (27).

$$G_1 = \tilde{G}(0) = -\text{diag}\{\lambda_i + \theta_i\}$$

Equation (27)

$\tilde{R}_i$ and $\tilde{R}_2$ can be established by using Eqs. (7) and (9), respectively.

$$\tilde{R}_i = \tilde{R}_i(0) = -\text{diag}\{\lambda_i + \theta_i\}$$

Equation (28)

$$\tilde{R}_2 = \frac{1}{2} \text{diag}\{\frac{(\lambda_i + \theta_i)(2\lambda_i + \theta_i)}{2} - \lambda_i \theta_i\}$$

Equation (29)
Substituting Eqs. (26) and (28) into Eq. (13), and rearranging it, we obtain an analytical tuning rule of \( \tilde{k}_i \) for a set of SISO controllers (or a multi-loop controller for non-interacting case) as Eq. (30).

\[
\tilde{k}_i = \text{diag} \left\{ \frac{1}{K_y(\lambda_i + \theta_s)} \right\} \tag{30}
\]

Then, an analytical tuning rule for the integral gain \( \tilde{K}_i \) for a multi-loop controller is derived by substituting Eq. (30) into Eq. (16) as follows in Eq. (31).

\[
\tilde{K}_i = \text{diag} \left\{ \frac{\Lambda_i}{K_y(\lambda_i + \theta_s)} \right\} \tag{31}
\]

Substituting Eqs. (26) to (29) into Eq. (20), an analytical tuning rule of \( \tilde{k}_c \) for a set of SISO controllers (or a multi-loop controller for non-interacting case) can be found as Eq. (32).

\[
\tilde{k}_c = \text{diag} \left\{ \frac{(\tau_c + \theta_s) \left[ 1 - \frac{\theta_s(2\lambda_i + \theta_s)}{2(\lambda_i + \theta_s)(\tau_c + \theta_s)} \right]}{K_y(\lambda_i + \theta_s)} \right\} \tag{32}
\]

In addition, the closed-loop dynamic interaction matrix \( \Phi \) can be found by rearranging Eq. (23) as Eq. (33).

\[
\Phi = \text{diag} \left\{ \frac{(\Lambda_i - \varphi_i)}{(1 - \varphi_i)} \right\} \tag{33}
\]

where \( \varphi_i \) is a dynamic factor consisting of the process parameters and closed-loop time constants, given by Eq. (34).

\[
\varphi_i = \frac{\theta_s(2\lambda_i + \theta_s)}{2(\lambda_i + \theta_s)(\tau_c + \theta_s)} \tag{34}
\]

Then, an analytical tuning rule for the proportional gain \( \tilde{K}_c \) for a multi-loop PI controller is obtained by substituting Eqs. (32) and (33) into Eq. (22) as follows in Eq. (35).

\[
\tilde{K}_c = \text{diag} \left\{ \frac{(\tau_c + \theta_s) \left[ \Lambda_i - \frac{\theta_s(2\lambda_i + \theta_s)}{2(\lambda_i + \theta_s)(\tau_c + \theta_s)} \right]}{K_y(\lambda_i + \theta_s)} \right\} \tag{35}
\]

For a multivariable process with SOPDT dynamics, the aforementioned procedure can also be applied to derive the analytical tuning rules. All resulting tuning rules for the processes of FOPDT and SOPDT dynamics are summarized in Table 1.

<table>
<thead>
<tr>
<th>Process</th>
<th>FOPDT</th>
<th>SOPDT</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_{i}e^{-\theta_s} )</td>
<td>( K_{i}e^{-\theta_s} )</td>
<td>( (\tau_s + 1)K_{i}e^{-\theta_s} )</td>
</tr>
<tr>
<td>( \theta_s(2\lambda_i + \theta_s) )</td>
<td>( \theta_s(2\lambda_i + \theta_s) )</td>
<td>( \theta_s(2\lambda_i + \theta_s) )</td>
</tr>
<tr>
<td>( \frac{\Lambda_i}{K_y(\lambda_i + \theta_s)} )</td>
<td>( \frac{\Lambda_i}{K_y(\lambda_i + \theta_s)} )</td>
<td>( \frac{\Lambda_i}{K_y(\lambda_i + \theta_s)} )</td>
</tr>
</tbody>
</table>

3. Multi-Loop Considerations

RGA analysis is widely employed as a measure of process interactions. However, it utilizes only steady-state information, which may cause a misleading result in multi-loop consideration. This potential problem can be
mitigated by considering $\Phi$ as the dynamic information in the design of the multi-loop control system. It is clear from Eqs. (33) and (34) that the property features of $\Phi$ are similar to those of $\Lambda$. The following relations and dependencies can be easily verified as Eqs. (36a) to (36e).

$$0 < \varphi_i < 1, \quad \forall \theta_i, \tau_i, \text{ and } \lambda_i$$  (36a)

$$\Phi_i = 1 \iff \Lambda_i = 1$$  (36b)

$$\Phi_i > 1 \iff \Lambda_i > 1$$  (36c)

$$\Phi_i < 1 \iff \varphi_i < \Lambda_i < 1$$  (36d)

$$\Phi_i < 0 \iff \varphi_i < \Lambda_i < 0$$  (36e)

Several different interaction modes are investigated based on different combinations of $\Phi_i$, $\Lambda_i$, and $\varphi_i$:

Case 1: $\Phi_i = 1 \iff \Lambda_i = 1$

In this ideal situation, it allows from Eq. (17) that opening or closing other loops have no effect on the $i$-th loop. There is no transmission interaction. However, it does not preclude the possibility that the manipulated variable might affect another controlled variable as one-way interaction. Hence, except for the one-way interaction, the tuning rules for a SISO controller can be directly applied to the design of the multi-loop PI controller.

Case 2: $\Phi_i > 1 \iff \Lambda_i > 1$

It is obvious from Eq. (17) that the open-loop gain is larger than the closed-loop gain and that the retaliatory effects from other loops act in opposition to the main effect of the $i$-th loop. Thus, closing other loops reduces the gain at the $i$-th loop. Moreover, it can be indicated from Eqs. (16) and (22) that the controller gain for the $i$-th loop, when closing other loops, has to be larger than that when opening other loops.

Case 3: $0 < \Phi_i < 1 \iff \varphi_i < \Lambda_i < 1$

In this situation, the closed-loop gain is larger than the open-loop gain for the $i$-th loop. When other loops are closed, the gain for the $i$-th loop will be increased and thus the effects from other loops are in the same direction as the main effect of the $i$-th loop.

It is well-known that the multi-loop control approach is usually confined to processes with modest interactions because of its limitation by controller structure. It should be noted that the proposed tuning method was also targeted for processes with modest interactions. The proposed tuning rules may not provide good performance for processes with significant interactions, such as the cases of $\Phi_i >> 1 \iff \Lambda_i >> 1$ and $\Phi_i << 1 \iff \Lambda_i \approx \varphi_i$. The cases where opening or closing other loops had an adverse effect on the controller gain ($\Phi_i < 0$ or $\Lambda_i < 0$) are also out of the scope of this study.

4. Performance Index and Maximum Sensitivity Criterion

In this paper, the performance and robustness of the multi-loop control system are evaluated by the following indices.

4.1 Integral absolute error index

To evaluate closed-loop performance, the integral absolute error (IAE) criterion is considered, which is defined as Eq. (37).

$$\text{IAE} = \int_0^\infty |e(t)| \, dt$$  (37)

where the error signal $e(t)$ denotes the difference between the set-point and controlled variable. The IAE should be as small as possible.

4.2 Maximum sensitivity (Ms) criterion

Ms tuning is one of the most well-known frequency-domain methods, in which the Ms values are related to the resonant peak of the sensitivity function. The robust stability of the control system can be measured by the magnitude of the Ms value. Skogestad and Postlethwaite (1996) introduced Ms as a tool for measuring system robustness. Aström et al. (1998) suggested that the desirable values of Ms for SISO systems are in the range of 1.2 to 2.

In the multi-loop systems, the sensitivity frequency response can be found by setting $s = j\omega$ in Eq. (1) as Eq. (38).

$$S(j\omega, \lambda) = [I + G(j\omega, \lambda)G_c(j\omega, \lambda)]^{-1}$$  (38)

According to the multivariable Nyquist stability theory, the maximum sensitivity function of the diagonal elements has a geometric property that is the inverse of the shortest distance from the critical point $(-1, 0)$ to the locus of the open-loop transfer function matrix: $G_c(j\omega)G(j\omega)$. Hence, smaller Ms values give the control system better robustness.

The maximum sensitivity Ms is obtained as follows with Eq. (39).

$$\text{Ms} = \max_{\omega \in \text{range}} |S(j\omega, \lambda)|$$  (39)

Ms can be expressed in the matrix form as Eq. (40).

$$\text{Ms} = \{\text{Ms}_i\} = \begin{bmatrix} \text{Ms}_{11} & \text{Ms}_{12} & \cdots & \text{Ms}_{1n} \\ \text{Ms}_{21} & \text{Ms}_{22} & \cdots & \text{Ms}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \text{Ms}_{n1} & \text{Ms}_{n2} & \cdots & \text{Ms}_{nn} \end{bmatrix}$$  (40)

Truong and Lee (2008) suggested the multi-loop Ms criterion to improve the robust performance and stability of the multi-loop control system for a regulatory problem. The multi-loop control system can also be made to meet the robustness bounds by finding the optimal closed-loop time constant, $\lambda_i$. This optimization problem in the frequency-domain can be described as Eq. (41).
\[
\min \left( \sum _{i} M_{s_{\min}} \right) \\
\text{s.t. } M_{s} \geq M_{s_{\min}} > 1
\]

where \( M_{s_{\min}} \) is the lower bound of the diagonal \( M_{s} \). Based on the relationship between the stability margin and maximum sensitivity function presented by Skogestad and Postlethwaite (1996), the minimum achievable \( M_{s_{\min}} \) should be greater than 1 for the sake of the maximum stability margin of the multi-loop control system. Figure 2 shows an example of the multivariable Nyquist plot of the sensitivity function for the Vinante and Luyben (VL) column (Luyben, 1986). As seen from Figure 2, the impact of \( M_{s_{\min}} \) on the overall closed-loop performance and robustness can be described as follows: at lower values of \( M_{s_{\min}} \), the multi-loop control system exhibits increased robust stability; however, the closed-loop response becomes sluggish (the integral absolute error values are larger). Conversely, the multi-loop control system exhibits better performance with less robust stability. The extended study herein showed that the desirable values of \( M_{s_{\min}} \) lie between 1.4 and 1.9, due to the tradeoffs between performance and robustness.

5. Simulation Study

In this section, several commonly used examples of industrial processes are considered to demonstrate the performance of the proposed method in comparison with those of other well-known methods. The IAE is used as the performance metric for the evaluation of controller performance. In the proposed method, the multi-loop Ms criterion is used for finding the optimal closed-loop time constant \( \lambda \).

**Example 1. Vinante–Luyben (VL) distillation column**

Luyben (1986) introduced the 24-tray tower, separating a mixture of methanol and water represented by the following transfer function matrix of Eq. (42).

\[
G(s) = \begin{bmatrix}
-2.2e^{-s} & 1.3e^{-0.3s} \\
7s + 1 & 7s + 1 \\
-2.8e^{-1.8s} & 4.3e^{-0.35s} \\
9.5s + 1 & 9.2s + 1
\end{bmatrix}
\]

The RGA, the closed-loop dynamic interaction matrix, and the dynamic factor are obtained from Eqs. (17), (33), and (34), respectively.

\[
\tilde{\Lambda} = \begin{bmatrix}
1.625 & 0 \\
0 & 1.625
\end{bmatrix}, \quad \Phi = \begin{bmatrix}
1.698 & 0 \\
0 & 1.645
\end{bmatrix}
\]

\[
\tilde{\Phi} = \begin{bmatrix}
0.104 & 0 \\
0 & 0.029
\end{bmatrix}
\]

The values of \( \tilde{\Lambda} \) and \( \Phi \) show that the interaction is modest. The multi-loop Ms criterion was used to find the optimum values of \( \lambda \), wherein \( M_{s_{\min}} \) was set to 1.8. As a result, the optimal values of \( \lambda \) were found to be 1.98 and 0.58 for loops 1 and 2, respectively. The performance of the proposed method was compared with those by the well-known design methods, such as BLT (Luyben, 1986), SAT (Loh et al., 1993), and the method of Lee et al. (1998). In the simulation study, the unit step changes in the set-point were sequentially made to loops 1 and 2. The resulting performance indices and controller parameters are tabulated in Table 2. Figure 3 compares the closed-loop responses afforded by the proposed method with those given by all the above-mentioned methods. It is apparent from the table and figure that the proposed controller provides performance superior to the other methods.

**Example 2. Industrial-scale polymerization (ISP) reactor**

The transfer function matrix for an ISP reactor system was studied by Chien et al. (1999) using Eq. (43).
It is a slow dynamic process with a time scale in hours. Following the same procedure, the RGA, the closed-loop dynamic interaction matrix, and the dynamic factor are given respectively by:

$$G(s) = \begin{bmatrix} 22.89e^{-0.2s} & -11.64e^{-0.4s} \\ 4.572s + 1 & 1.807s + 1 \\ 4.689e^{-0.2s} & 5.8e^{-0.4s} \\ 2.174s + 1 & 1.801s + 1 \end{bmatrix}$$

(43)

$$\tilde{A} = \begin{bmatrix} 0.709 & 0 \\ 0 & 0.709 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0.699 & 0 \\ 0 & 0.653 \end{bmatrix}$$

and $$\tilde{\Phi} = \begin{bmatrix} 0.030 & 0 \\ 0 & 0.161 \end{bmatrix}$$

Table 2  Controller parameters and resulting performance indices for example 1

<table>
<thead>
<tr>
<th>Tuning method</th>
<th>Loop</th>
<th>$K_c$</th>
<th>$\tau_n$</th>
<th>IAE$_i$</th>
<th>IAE$_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed</td>
<td>1</td>
<td>-1.86</td>
<td>7.49</td>
<td>2.98</td>
<td>1.18</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.81</td>
<td>9.38</td>
<td>0.95</td>
<td>1.10</td>
</tr>
<tr>
<td>BLT</td>
<td>1</td>
<td>-1.07</td>
<td>7.10</td>
<td>4.9</td>
<td>0.79</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.97</td>
<td>2.58</td>
<td>1.59</td>
<td>1.33</td>
</tr>
<tr>
<td>SAT</td>
<td>1</td>
<td>-1.35</td>
<td>3.00</td>
<td>3.50</td>
<td>0.67</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.36</td>
<td>1.33</td>
<td>1.46</td>
<td>1.52</td>
</tr>
<tr>
<td>Lee et al. (1998)</td>
<td>1</td>
<td>-1.31</td>
<td>2.26</td>
<td>3.48</td>
<td>0.81</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.97</td>
<td>2.42</td>
<td>1.47</td>
<td>1.22</td>
</tr>
</tbody>
</table>

IAE$_i$: IAE for the step change in loop $i$
IAE$_s$: total sum of each IAE$_i$

Fig. 3  Closed-loop time responses to the sequential step changes in the set-point for example 1
Ms$_{\text{nom}}$ was set to 1.4, and $\lambda_i$ were obtained as 0.148 and 1.37 for loops 1 and 2, respectively. The proposed, BLT (Luyben, 1986), SAT (Loh et al., 1993), and the method of Chien et al. (1999) were used to design the multi-loop PI controller. Figure 4 compares the closed-loop responses by the proposed controller and those by the others, where the sequential unit step changes in the set-points were introduced at $t = 0$ and $t = 15$. It is evident from this figure that the proposed controller affords superior closed-loop performance with fast and well-balanced responses over the others. The resulting controller parameters, together with the performance indices, are also given in Table 3. It can be seen that the closed-loop response by the proposed controller brought the smallest total IAE among all comparative methods.

### Example 3. Ogunnaike and Ray (OR) distillation column

A well-known multi-product distillation column for the separation of a binary ethanol–water mixture studied by Ogunnaike et al. (1983) is considered. The transfer function matrix is given by Eq. (44).

<table>
<thead>
<tr>
<th>Table 3</th>
<th>Controller parameters and resulting performance indices for example 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tuning method</td>
<td>Loop</td>
</tr>
<tr>
<td>Proposed</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td>BLT</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td>SAT</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Chien et al. (1999)</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
</tbody>
</table>

![Fig. 4](image_url) Closed-loop time responses to the sequential step changes in the set-point for example 2
The RGA, the closed-loop dynamic interaction matrix, and the dynamic factor are found by:

\[
\tilde{G} = \begin{bmatrix}
0.66e^{-2.6s} & -0.61e^{-3.5s} & -0.0049e^{-s}
\end{bmatrix}
\]

\[
\begin{bmatrix}
6.7s + 1
8.64s + 1
9.06s + 1
\end{bmatrix}
\]

\[
\tilde{\Lambda} = \begin{bmatrix}
1.11e^{-6.5s} & -2.36e^{-3s} & -0.01e^{-1.2s}
\end{bmatrix}
\]

\[
\begin{bmatrix}
3.25s + 1
5s + 1
7.09s + 1
\end{bmatrix}
\]

\[
\tilde{\phi} = \begin{bmatrix}
-34.68e^{-0.2s} & 46.2e^{-0.4s} & 0.87(11.61s + 1)e^{-s}
\end{bmatrix}
\]

\[
\begin{bmatrix}
8.15s + 1
10.9s + 1
(3.89s + 1)(18.8s + 1)
\end{bmatrix}
\]

\[
(44)
\]

Fig. 5  Closed-loop time responses to the sequential step changes in the set-point for example 3

In the simulation study, \(M_{slow} = 1.8\), and \(\lambda_i\) were found to be 13.65, 13.50, and 2.85 for loops 1, 2, and 3, respectively. Figure 5 shows the closed-loop responses provided by various multi-loop PI controllers, where the magnitudes of the sequential step changes in the set-points of loops 1, 2, and 3 were 1, 2, and 15, respectively. The resulting performance indices and controller parameters
Table 4  Controller parameters and resulting performance indices for example 3

<table>
<thead>
<tr>
<th>Tuning method</th>
<th>Loop</th>
<th>$K_C$</th>
<th>$\tau$</th>
<th>$IAE_i$</th>
<th>$IAE_{ii}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed</td>
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<td>1.49</td>
<td>8.09</td>
<td>16.91</td>
<td>25.73</td>
</tr>
<tr>
<td></td>
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<td>6.50</td>
<td>6.47</td>
<td>33.61</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4.85</td>
<td>11.48</td>
<td>0.87</td>
<td>1.81</td>
</tr>
<tr>
<td>BLT</td>
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<td>1.51</td>
<td>16.4</td>
<td>33.04</td>
<td>73.35</td>
</tr>
<tr>
<td></td>
<td>2</td>
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<td>18.0</td>
<td>12.62</td>
<td>95.90</td>
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<tr>
<td></td>
<td>3</td>
<td>2.63</td>
<td>6.61</td>
<td>1.70</td>
<td>2.70</td>
</tr>
<tr>
<td>SAT</td>
<td>1</td>
<td>2.71</td>
<td>7.44</td>
<td>18.79</td>
<td>62.94</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.37</td>
<td>10.52</td>
<td>4.92</td>
<td>44.94</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4.56</td>
<td>3.09</td>
<td>1.87</td>
<td>7.07</td>
</tr>
</tbody>
</table>

are summarized in Table 4. The proposed controller provides superior performance over the BLT (Luyben, 1986) and SAT (Loh et al., 1993).

Example 4. Alatiqui case 1 (A1) column

The transfer function matrix model for the A1 column was introduced by Luyben (1986) as follows with Eq. (45).

$$G(s) = \begin{bmatrix}
-2.23e^{-5s} & -2.94(7.9s + 1)e^{-0.05s} & 0.017e^{-0.2s} & -0.64e^{-20s} \\
(36s + 1)(25s + 1) & (23.7s + 1)^2 & (31.6s + 1)(7s + 1) & (29s + 1)^2 \\
-2.23e^{-5s} & 3.46e^{-10s} & -0.51e^{-7s} & 1.68e^{-2s} \\
(35s + 1)^2 & (32s + 1) & (32s + 1)^2 & (28s + 1)^2 \\
-1.06e^{-22s} & 3.51e^{-13s} & 4.41e^{-10s} & -5.38e^{-5s} \\
(17s + 1)^2 & (12s + 1)^2 & (16.2s + 1) & (17s + 1) \\
-5.73e^{-25s} & 4.32(25s + 1)e^{-0.01s} & -1.25e^{-2.8s} & 4.78e^{-1.15s} \\
(50s + 1)(8s + 1) & (50s + 1)(5s + 1) & (43.6s + 1)(9s + 1) & (48s + 1)(5s + 1)
\end{bmatrix}$$

The RGA, the closed-loop dynamic interaction matrix, and the dynamic factor are found by:

$$\hat{\Lambda} = \begin{bmatrix} 3.53 & 0 & 0 & 0 \\
0 & 2.77 & 0 & 0 \\
0 & 2.26 & 0 & 1.48 \\
0 & 0 & 0 & 1.48 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 3.63 & 0 & 2.82 & 0 \\
0 & 0 & 2.33 & 0 \\
0 & 0 & 0 & 1.49 \end{bmatrix}, \quad \text{and } \hat{\varphi} = \begin{bmatrix} 0.039 & 0 & 0 & 0 \\
0 & 0.028 & 0 & 0 \\
0 & 0 & 0.051 & 0 \\
0 & 0 & 0 & 0.015 \end{bmatrix}$$

$M_{s_{\text{low}}}$ was set to 1.8, and $\lambda_i$ were found to be 59.57, 4.00, 2.85, and 0.92 for loops 1, 2, 3, and 4, respectively. The proposed, BLT (Luyben, 1986), and the method of Lee et al. (1998) were used to design the multi-loop PI controller. In the simulation study, the unit step changes in the set-point were sequentially made to loops 1, 2, 3, and 4. Figure 6 compares the closed-loop responses afforded by the proposed method with those by the others for a unit step change in the set-point of $Y_i$ (at $t = 0$). The resulting performance indices and controller parameters for all of the above-mentioned methods are listed in Table 5. The results confirm the noteworthy performance improvement of the proposed controller.

Conclusions

By extending the Maclaurin series approach to the multi-loop control system, controller gains of the interacting multi-loop system were expressed in terms of those of the non-interacting system and the interaction factor. In particular, the integral gain of the multi-loop controller was designed by a simple multiplication of a single-loop controller and a well-known RGA. The closed-loop dynamic interaction matrix was introduced to relate the proportional gain of the multi-loop controller to that of a single-loop controller. Based on these relations, the analytical tuning rules for multi-loop PI controllers were derived for the FOPDT and the SOPDT processes. The multi-loop Ms criterion was employed to enhance both the performance and robust-
ness of the proposed control system. A variety of illustrative examples taken from previous works were investigated to demonstrate the effectiveness of the proposed method. The simulation results confirmed that the proposed method afforded superior performance over several well-known existing methods.

Acknowledgment

This research was supported by KOSEF research grants in 2009.
Table 5  Controller parameters and resulting performance indices for example 4

<table>
<thead>
<tr>
<th>Tuning method</th>
<th>Loop</th>
<th>$K_C$</th>
<th>$\tau$</th>
<th>IAE $\xi$</th>
<th>IAE $\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed 1</td>
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<td>62.81</td>
<td>66.41</td>
<td>2.91</td>
<td>31.22</td>
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<td>61.24</td>
<td>6.32</td>
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<td></td>
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</tr>
<tr>
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<td>29.95</td>
</tr>
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<td>BLT 1</td>
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<td>74.03</td>
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<td>134.80</td>
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<td>0.15</td>
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<td></td>
<td>0.73</td>
<td>36.93</td>
<td>15.99</td>
<td>0.89</td>
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</table>

Literature Cited