Analytical design of fractional-order proportional-integral controllers for time-delay processes

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**Abstract**

A new design method of fractional-order proportional-integral controllers is proposed based on fractional calculus and Bode’s ideal transfer function for a first-order-plus-dead-time process model. It can be extended to be applied to various dynamic models. Tuning rules were analytically derived to cope with both set-point tracking and disturbance rejection problems. Simulations of a broad range of processes are reported, with each simulated controller being tuned to have a similar degree of robustness in terms of resonant peak to other reported controllers. The proposed controller consistently showed improved performance over other similar controllers and established integer PI controllers.

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**1. Introduction**

Fractional-order dynamic systems are useful in presenting various stable physical phenomena with anomalous decay [1]. Fractional calculus (i.e. fractional integro-differential operators) is a generalization of integration and differentiation to non-integer orders. It is obtained from ordinary calculus by extending ordinary differential equations (ODE) to fractional-order differential equations (FODE). Similarly, a fractional-order proportional-integral-derivative (FOPID) controller is a generalization of a standard (integer) PID controller; its output is a linear combination of the input and the fractional integral or derivative of the error [2]. It affords more flexibility in PID controller design due to its five controller parameters (instead of the standard three): proportional gain, integral gain, derivative gain, integral order, and derivative order. However, the tuning rules of fractional-order PID (FOPID) controllers are much more complex than those of standard (integer) PID controllers with only three parameters. Several design methodologies of FOPID controllers have been introduced to facilitate their use: Bode first reported fractional structures in feedback-loops [3,4]. This was extended by Barbosa et al. [5], who reported a feedback amplifier obtained by considering a feedback-loop in terms of the performance of a closed-loop that was invariant to changes of amplifier gain. However, this concept was not rigorously developed and remained neglected for decades. Oustaloup [6] introduced fractional-order algorithms for the control of dynamic systems based on non-integer derivatives and demonstrated significant improvement of CRONE (Commande Robuste d’Ordre Non Entier) controllers over integer PID controllers. CRONE controllers are obtained using a rational form and the major differences between the three generations lie in the design of the open-loop, the slope of which depends on consideration of plant uncertainty. The generalized PID controller, PP\textsuperscript{D}, that involves a fractional-order integrator (λ) and a fractional-order differentiator (μ), was suggested by Podlubny [7]. The two extra parameters (λ and μ) give this type of controller improved flexibility over integer PID controllers, giving it much industrial applicability [8,9]. Tuning methods of PP\textsuperscript{D} controllers can be generally classified as either analytic or heuristic [10,11].

Most analytical methods are tuned by considering the non-linear objective function, which is dependent on user-imposed specifications [5, 11-14]. Barbosa et al. [5] introduced the tuning of integer PID controller by considering the system similar to a fractional-order system that is done by using the ISE error minimization. Monje et al. [13] proposed five conditions taking into account phase and gain margins specifications, as well as the constraints over the sensitivity functions. Valério and Da Costa [11] considered Ziegler-Nichols-type tuning rules for the first order plus time delay (FOPDT) processes. In addition, F-MIGO (i.e., peak sensitivity constrained integral gain optimization for the fractional-order PI control system) method [14] was developed for the FOPDT class of dynamic systems, which is generalized to...
handle the FOPI control system from the so-called MICO (i.e., Ms, the maximum sensitivity constrained integral gain optimization) developed by Åström et al. [15–17]. In accordance with the F-MICO method, the Nyqvist plot of the open-loop transfer function lies outside the circle such that it encloses both the Ms and Mp circles. This circle has the center and radius as shown in [14]. Furthermore, Vinagre [18] suggested setting $\lambda = \mu$ and imposing a phase margin at gain crossover frequency. Caponetto et al. [19] proposed a method selecting freely a $\lambda = \mu > 1$, which is allowed freely choosing controller parameters by imposing a phase margin at gain crossover frequency. A fractional PI controller was tuned by the combination of gain and phase margin requirements with a flat phase for the open-loop at critical frequency introduced by Chen et al. [20]. The internal model control (IMC) methodology can be also used in some cases to obtain PID or fractional PID controllers [11].

This work proposes a new analytic method of FOPI controller design for enhanced set-point tracking and disturbance rejection responses of processes with time delays. It is based largely on fractional calculus and Bode’s ideal transfer function. By using frequency domain, the proposed FOPI tuning rules can be derived for first-order-plus-dead-time (FOPDT) models and can be applied to various process models.

The paper is organized as follows. The fundamentals of fractional calculus and their application for obtaining the FOPI controller are given in Section 2. In Section 3, the generalized FOPI controller tuning rules and some important robustness and performance indices are introduced. Section 4 gives some illustrated examples, where a comparison with other design methods is presented. Some important guidelines are introduced in Section 5. Conclusions are given in Section 6.

2. Preliminaries

This section introduces some fundamentals of fractional calculus, the problem statement required to understand fractional systems, and the examined controller.

2.1. Fractional calculus

Fractional calculus [21] is a generalization of ordinary calculus. It develops a functional operator, $D_{a}^{\lambda}$, associated to the order of an operation $v$ (where $\Re v$) not restricted to integers that generalizes usual derivatives (for positive $v$) and integrals (for negative $v$). There are various definitions of fractional differentiation. However, the most commonly used is the Riemann–Liouville definition [9,21], which is generalized by two equalities easily proven for integer orders:

$$D_{a}^{\lambda}f(t) = \frac{1}{\Gamma(n-v)} \frac{d^n}{dt^n} \int_{a}^{t} f(\tau) (t-\tau)^{-n+v} d\tau, n-1 < v < n$$

(1)

where $\Gamma(\bullet)$ denotes Euler’s gamma function. $a$ and $t$ are the limits.

Note that the Laplace transform of the fractional derivative/integral in (1) follows the rule for zero initial condition for order $v$ ($0 < v < 1$):

$$L(D_{a}^{\lambda}f(t)) = s^{\lambda}F(s)$$

(2)

The initial conditions imply that a dynamic system described by differential equations involving fractional derivatives gives rise to transfer functions with fractional powers of $s$. This is described further elsewhere [21].

2.2. Integer order approximation

For fractional-order controllers to be used for simulation and hardware applications with transfer functions that involve fractional orders of $s$, the transfer function should be approximated as an integer-order transfer function with similar behavior, which includes an infinite number of poles and zeros. Nevertheless, reasonable approximations can be obtained with finite numbers of poles and zeros. In this case, the Oustaloup continuous integer-order approximation [6] based on the recursive distribution of poles and zeros is employed here:

$$s^\lambda \approx k \prod_{n=1}^{N} \left( 1 + \frac{s}{\omega_{p,n}} \right)$$

(3)

Eq. (3) is valid over the frequency range $[\omega_{h}, \omega_{l}]$, where the gain, $k$, should be adjusted for both sides of (2) to have unity gain at the gain crossover frequency, $s^\lambda$ (i.e., $\omega_{c} = 1$ rad/s). Eight poles and zeros (i.e., $N=8$) is chosen, since $\omega_{h}$ and $\omega_{l}$ are respectively $0.001\omega_{c}$ and $1000\omega_{c}$. It is important to note that low values result in simpler approximations, but may cause ripples in both gain and phase behaviors. The ripples can be functionally neglected by increasing $N$, and hence also increasing computation costs. In addition, frequencies of zeroes and poles in (3) are given as follows:

$$\omega_{h,l} = \omega_{c}\sqrt{\eta}$$

(4a)

$$\omega_{p,n} = \omega_{a,n} \alpha, \quad n = 1, 2, \ldots N$$

(4b)

$$\omega_{p,n+1} = \omega_{a,n} \eta, \quad n = 1, 2, \ldots N-1$$

(4c)

$$\alpha = (\omega_{h}/\omega_{l})^{N/4}$$

(4d)

$$\eta = (\omega_{h}/\omega_{l})^{(1-v)/N}$$

(4e)

It can be dealt with inverting (3) when $v < 0$. But in the case $|v| > 1$, the approximation will be unsatisfactory. Therefore, it is common to split fractional power of $s$ as follows:

$$s^\lambda = s^n s^{\lambda-n}, \quad v = n + \delta \land n \in \mathbb{Z} \land \delta \in [0; 1]$$

(5)

2.3. FOPI controller

Fractional calculus gives the fractional integro-differential equation of a FOPI controller as:

$$u(t) = K_{c}e(t) + K_{d}D^{-\lambda}_{a}e(t), \quad (\lambda > 0)$$

(6)

where $K_{c}$ and $K_{d}$ represent the proportional and integral terms of the FOPI controller, respectively. $\lambda$ is the fractional order of the integral.

The continuous transfer function of the FOPI controller can be obtained by Laplace transformation:

$$G_{c}(s) = K_{c} + \frac{K_{1}}{s^{\lambda}}$$

(7)

The FOPI controller has three parameters ($K_{c}, K_{1}$, and $\lambda$) to tune, since the fractional order $\lambda$ is not necessarily integer. An integer PI controller is a special case of this FOPI controller where $\lambda = 1$. This expansion provides more flexibility in achieving control objectives. However, it is often complicated by requiring a non-linear objective function and user-defined constraints to obtain controller parameters that satisfy some specified performance criterion.

By substituting $s = j\omega$ into (7), the FOPI controller is represented in the frequency domain as:

$$G_{c}(j\omega) = K_{c} + \frac{K_{1}}{j^{\lambda}}$$

(8)

The fractional power of $j\omega$ can be written as

$$j^{\lambda} = \omega^{\lambda} = \cos(\pi\lambda/2) + j \sin(\pi\lambda/2)$$

(9)

where $n = 0, \pm (1/\lambda), \pm (2/\lambda), \ldots, \pm (m/\lambda)$. Therefore, the following convenient form is obtained:

$$j^{\lambda} = \omega^{\lambda}(\cos(\pi\lambda/2) + j \sin(\pi\lambda/2))$$

(10)
Substituting (10) into (8) and rearranging gives a complex equation for the FOPI controller:

\[
G_C(j\omega) = \left( K_c + \frac{K_1 \cos \gamma_1}{\tau s} \right) - j \left( \frac{K_1 \sin \gamma_1}{\tau s} \right)
\]  

(11)

Consider the standard block diagrams of the feedback control strategies in Fig. 1. \(G(s)\)and \(G_c(s)\) denote the process and feedback controller, respectively. \(y(s), r(s), d(s),\) and \(u(s)\) correspond to the controlled output, the set-point input, the disturbance input, and the manipulated variables, respectively.

The closed-loop transfer function for set-point changes is:

\[
y(s)\over r(s) = \frac{G(s)G_c(s)}{1 + G(s)G_c(s)}
\]  

(12)

Rearranging and solving for \(G_c(s)\) gives an expression for the feedback controller:

\[
G_c(s) = \frac{1}{G(s)} \left( \frac{y(s)}{r(s)} \right)
\]  

(13)

In direct synthesis, analytical tuning rules for the feedback controller can be derived employing two key assumptions; the process model \(G(s)\) is available and the desired closed-loop response, \(\frac{y(s)}{r(s)}\)\(_d\), can be expressed as a closed-loop transfer function for set-point changes. Consequently, the ideal feedback controller, \(G_c(s)\), can be derived by replacing the unknown \(\frac{y(s)}{r(s)}\) and \(G(s)\) by \(\frac{y(s)}{r(s)}\)\(_d\) and \(G(s)\), respectively.

\[
G_c(s) = \frac{1}{G(s)} \left( \frac{y(s)}{r(s)} \right)_{d}
\]  

(14)

\(\frac{y(s)}{r(s)}\)\(_d\) should be chosen so that the resulting controller is physically realizable. Since the feedback controller in (14) does not have the standard form of a FOPI controller, it is necessary to find a FOPI controller that closely matches the ideal feedback controller.

3. Analytical design of generalized FOPI controller tuning rules

The ideal open-loop transfer function for a delay-free process, based on the feedback amplifiers of Bode [4], is defined as:

\[
L(s) = \left( \frac{\omega_{\phi} s}{\tau s + 1} \right)^{\lambda}, \quad \gamma = \frac{\omega_{\phi}}{\omega_{\phi}}
\]  

(15)

where \(\omega_{\phi}\) is the gain crossover frequency (i.e., \(|L(j\omega_{\phi})| = 1\)). \(\gamma\) represents the slope of the magnitude curve, which may assume non-integer values. \(L(s)\) is a fractional-order differentiator for \(\gamma < 0\) and a fractional-order integrator for \(\gamma > 0\). Bode diagrams of \(L(s)\) in the range of \(1 < \gamma < 2\) have important characteristics, e.g. the magnitude curve has a constant slope of \(-20\gamma\) dB/dec. The phase plot is horizontal at \(-\pi/2\) rad. The Nyquist plot is linear from the origin with argument \(-\pi/2\).

The desired closed-loop transfer function of the fractional-order system given by the unit feedback system in Fig. 2 can be expressed as:

\[
\frac{y(s)}{r(s)} = \frac{L(s)}{1 + L(s)} = \frac{1}{1 + (s/\omega_{\phi})^{\lambda}}
\]  

(16)

This choice of \(L(s)\) gives a closed-loop system with the desirable property of being insensitive to gain changes. The gain crossover frequency, \(\omega_{\phi}\), will vary depending on the changes of gain but the phase margin of system will always remain \(\phi_m = \pi(1 - \gamma)/2\) rad [5].

In this work, our aim is to design the desired closed-loop transfer function for the fractional-order processes with time delays. Therefore, the time delay should be taken into account in order to avoid the non-causality for the controller. The closed-loop transfer function of the fractional reference model is desirably chosen by [5]:

\[
\frac{y(s)}{r(s)} = \frac{L(s)}{1 + L(s)} e^{-\theta s} = \frac{e^{-\theta s}}{1 + (s/\omega_{\phi})^{\lambda}}
\]  

(17)

where \(\theta\) denotes the time delay.

Substituting (17) into (14) allows simplification of the feedback controller for the desired set-point changes:

\[
G_c(s) = \frac{1}{G(s)} \left[ \frac{e^{-\theta s}}{(s/\omega_{\phi})^{\lambda} + 1} - e^{-\theta s} \right]
\]  

(18)

It should be noted that the above desired closed-loop transfer function involves two parts: the desired closed-loop transfer function of delay-free fractional-order system given by (16) and delay part \((e^{-\theta s})\). It is implied from (18) that the delay portion (i.e. non-minimum phase) of \(G_c(s)\) is cancelled by the time delay term in the numerator. Therefore, the controller has no causality problem.

Previously reported test results [14] are used here to select the fractional-order of integral, \(\lambda\); the first being the set of systems chosen in [15], each of which are approximated by an FOPDT process model because it is widely used industrially. An FOPDT process model can be represented as:

\[
\tilde{G}(s) = \frac{K e^{-\theta s}}{s^\lambda + 1}
\]  

(19)

where \(K, \tau,\) and \(\theta\) denote the process's gain, time constant, and dead time, respectively. The process dynamics can be conveniently characterized by the relative dead time [14]:

\[
\Delta = \frac{\theta}{\tau + \theta}
\]  

(20)

where \(\Delta \in [0, 1]\) and represents a measure of the difficulty in controlling the process.

FOPDT processes with the dynamic characterized as \(\Delta \leq 0.6\) (i.e., \(\theta > \tau\)) are delay-dominated; those with \(\Delta < 0.1\) (i.e., \(\theta < \tau\)) are lag-dominated. FOPDT processes with \(\theta = \tau\) are lag and delay balanced plants. Based on the observation that the fractional-order of integral, \(\lambda\), depends on the value of \(\Delta\) and is almost invariant with respect to \(\theta\), the following guideline was proposed [22]:

\[
\lambda = \begin{cases} 
1.1, & \text{if } \Delta \geq 0.6 \\
1.0, & \text{if } 0.4 \leq \Delta < 0.6 \\
0.9, & \text{if } 0.1 \leq \Delta < 0.4 \\
0.7, & \text{if } \Delta < 0.1
\end{cases}
\]  

(21)

In the ambiguous region \(0.4 \leq \Delta < 0.6\), the best fractional order is close to unity, indicating that a fractional-order controller may be unnecessary. Delay-dominated processes require orders higher than 1; lag-dominated processes can be efficiently controlled by lower order controllers.
Substituting (19) into (18), gives the ideal feedback controller for FOPDT processes as:

$$G_C(s) = \frac{1}{\mathbb{R}} \left[ \frac{s^3 + 1}{(s + \omega_{cg})^3} e^{-s\tau} \right]$$

(22)

Substituting $s = j\omega$ into (22) allows the resulting FOPI controller to be represented in the frequency domain as:

$$G_C(j\omega) = \frac{1}{\mathbb{R}} \left[ \frac{1 + (\omega/\omega_{cg})^3 \cos \phi - \cos \phi + (\omega/\omega_{cg}) \sin \phi + \sin \phi}{(1 + (\omega/\omega_{cg})^3 \cos \phi - \cos \phi + (\omega/\omega_{cg}) \sin \phi + \sin \phi)^2} \right] \left[ 1 + (\omega/\omega_{cg})^3 \cos \phi - \cos \phi + (\omega/\omega_{cg}) \sin \phi + \sin \phi \right]^2$$

(23)

where,

$$e^{-s\tau} = \cos \phi - j\sin \phi, \phi = \omega\tau$$

(24)

$$\left(j\frac{\omega}{\omega_{cg}}\right)^3 = e^{j\pi/2} \omega^3 = \left(\frac{\omega}{\omega_{cg}}\right)^3 \cos \phi + j\left(\frac{\omega}{\omega_{cg}}\right)^3 \sin \phi, \phi = \omega\tau$$

(25)

Comparing (23) and (11) yields analytical tuning rules of the integral and proportional terms of the proposed FOPI controller, respectively:

$$K_I = \frac{\omega^3}{\mathbb{R}} \left[ 1 + \frac{(\omega/\omega_{cg})^3 \cos \phi - \cos \phi + (\omega/\omega_{cg}) \sin \phi + \sin \phi}{(1 + (\omega/\omega_{cg})^3 \cos \phi - \cos \phi + (\omega/\omega_{cg}) \sin \phi + \sin \phi)^2} \right] \left[ 1 + (\omega/\omega_{cg})^3 \cos \phi - \cos \phi + (\omega/\omega_{cg}) \sin \phi + \sin \phi \right]^2$$

(26a)

$$K_P = \frac{1}{\mathbb{R}} \left[ \frac{1 + (\omega/\omega_{cg})^3 \cos \phi - \cos \phi + (\omega/\omega_{cg}) \sin \phi + \sin \phi}{(1 + (\omega/\omega_{cg})^3 \cos \phi - \cos \phi + (\omega/\omega_{cg}) \sin \phi + \sin \phi)^2} \right] \frac{K_I \cos \phi}{\omega^3}$$

(26b)

In this paper, the fractional-order of integral ($\lambda$) is firstly selected based on (21). The parameters of proportional and integral gains are then obtained by adjusting the values of $\omega, \omega_{cg}$ and $\gamma$ to satisfy a given desirable resonant peak (Mp) value. It is clear that the set of $(\omega, \omega_{cg}, \gamma)$ is not unique to a given Mp value. The remaining degree of freedom is used to find the optimal set of $(\omega, \omega_{cg}, \gamma)$ for the best integral absolute error (IAE) value while it satisfies the Mp equality constraint. In addition, the values of $\omega, \omega_{cg}$, and $\gamma$ can be set up as suitable tradeoffs between performance and robustness in any given dynamic models.

It is important to note that IAE, overshoot, total variation (TV) and maximum sensitivity (Ms) value are used as performance indices for the comparisons, whereas Mp is utilized as design specification. The definitions of every index are given as follows:

### 3.3. Total variation (TV)

To evaluate the required control effort, TV is a good measure of the signal’s smoothness. It is calculated from the total variation of the manipulated variable by considering the sum of all moves in both directions:

$$TV = \sum_{i=1}^{\infty} |u_{i+1} - u_i|$$

(28)

It should also be as small as possible.

### 3.4. Maximum sensitivity (Ms) value

The Ms value is defined by

$$Ms = \max_{0 < \omega < \omega_{cg}, 1 < \gamma < 2} \left| \frac{1}{1 + G_C(j\omega)G(j\omega)} \right|$$

(29)

Note that Ms value is also another typical index for evaluating the robustness level and were also calculated for all examples. Since Ms is the inverse of the shortest distance from the Nyquist curve of the loop function to the critical point (−1,0), a small value indicates that the stability margin of the control system is large.

### 3.5. Resonant peak (Mp) criterion

Resonant peak, Mp, is defined as the maximum magnitude of the closed-loop frequency response, a good indicator of both the performance and the robustness of a control system [24,25].

$$M_p = \max_{0 < \omega < \omega_{cg}, 1 < \gamma < 2} \left| \frac{G_C(j\omega)G(j\omega)}{1 + G_C(j\omega)G(j\omega)} \right|$$

(30)

To ensure fair comparison, the proposed FOPI controllers are tuned by adjusting the ratio $\omega/\omega_{cg}$ and $\gamma$ so that Mp values are identical, i.e. each controller has the same or higher robustness compared with the other controllers in terms of the maximum magnitude of the complementary sensitivity function.

### 4. Simulation study

To assess the simplicity and effectiveness of the proposed FOPI tuning rules, several illustrative simulation examples are considered here.

#### 4.1. Example 1

The previously studied FOPDT process described by the transfer function [14,15]:

$$\hat{G}(s) = \frac{e^{-s}}{0.098s + 1}$$

(31)

This process has been shown to be well suited to the F-MIGO controller suggested by Monje et al. [14], demonstrated superiority over integer PI controllers, such as AMIGO controllers and the others [14]. In the simulation studies, the proposed FOPI controller is mainly compared with those of the F-MIGO and AMIGO methods, as well as the integer PI controller cascaded with a first-order lag filter suggested by Lee et al’s method [26]. For both the F-MIGO and proposed methods, the fractional-order of the integral, $\lambda$, is set as 1.1 (because $\Delta = 0.92$) based on (21). As one can be seen in Fig. 3, it is implied that the choices of $\lambda$ values affect the closed-loop performance at any given values of Mp for this delay-dominated process. It is also apparent from the figure that the best integral order ($\lambda$) is 1.1, which corresponds to the lowest value of IAE. This result is well matched with the guideline given in (21) in the case of the delay-dominated process.

To ensure fair comparison, the proposed FOPI controller was tuned to have the same robustness as the F-MIGO controller. Accordingly, $\omega/\omega_{cg} = 1.95/3.60$ and $\gamma = 1.001$ were selected to obtain $M_p = 1.037$.
for the proposed FOPI controller as same as that of the F-MIGO, which was clearly implied by considering the closed-loop frequency responses in Fig. 4.

The resulting controller parameters, together with calculated performances and robustness indices are listed in Table 1. Unit step changes in the set-point and load disturbances were introduced at \( t=0 \) (s) and \( t=15 \) (s), respectively. The corresponding simulation results are shown in Fig. 5, which shows that the proposed controller affords superior closed-loop performance with fast and well-balanced responses over those of the other controllers for both servo and regulatory problems. The controller output (manipulated variable) responses are also shown in Fig. 6, which is demonstrated that the control efforts of all comparative methods are smooth enough for the successful operation.

### 4.2. Example 2

The following high-order process model is considered here:

\[
\hat{G}(s) = \frac{e^{-15s}}{(s + 1)^3}
\]  

It can be reasonably approximated to the FOPDT process model as follows [14]:

\[
\hat{G}(s) = \frac{e^{-16.23s}}{1.76s + 1}
\]  

According to (21), the fractional-order of the integral, \( \lambda \), is chosen as 1.1 for this delay-dominated process, since \( \Delta \) is calculated as 0.90. For the proposed controller, \( \omega_c/\omega_{cg} = 0.12/0.135 \) and \( \gamma = 1.01 \) were selected to give \( M_p = 1.047 \), which is smaller than that of the F-MIGO controller \( (M_p = 1.095) \). Fig. 7 shows the closed-loop time responses to both disturbances and set-point changes. The proposed FOPI controller results in the fastest, i.e. lowest settling time, and the best-balanced response with lowest IAE values. The resulting controller parameters, together with the performance and robustness indices, are summarized in Table 2, which shows that the proposed method affords improved performance for both disturbance rejection and set-point tracking.
In addition, it is clear from the controller output responses (Fig. 8) and TV values (Table 2) that the proposed FOPI controller provides the small control effort.

4.3. Example 3

The following lag-dominated process model is considered for models' performance comparison [14]:

\[ G(s) = \frac{1}{(s + 1)(0.25 + 1)} \]  

(34)

It can be approximated to the FOPDT by [14]:

\[ G(s) = e^{-0.105s} \frac{1}{1.11s + 1} \]  

(35)

According to (21), the fractional-order of the integral, \( \lambda \), is chosen as 0.7 for this lag-dominated process, since \( \Delta = 0.09 \). To ensure fair comparison, the ratio \( \omega_c/\omega_c^g = 10/9.95 \) and parameter \( \gamma = 1.45 \) were selected to give more robustness than those of the other comparative methods, as \( M_p = 1.048 \).

Fig. 9 shows disturbance and set-point responses of the various methods, demonstrating the superiority of the proposed method. The characteristics summarized in Table 3 also confirm the significantly improved performance of the proposed method. The controller output responses are also shown in Fig. 10. The TV of the proposed method has the smallest value for the disturbance rejection and the biggest value for the set-point tracking but it is still smooth enough for a successful operation.

4.4. Example 4

Consider the FODT process model studied by Padula et al. [27]:

\[ G(s) = e^{-0.67s} \frac{1}{(s + 1)} \]  

(36)

It has balanced delay-lag dynamic because \( \Delta = 0.4 \). For this process, the fractional order is scanned in the range of \( 0.7 \leq \lambda \leq 1.3 \). The best controller is then picked based on the IAE criterion with a given desired \( M_p \) value as shown in Fig. 11. Therefore, the fractional-order, \( \lambda \), is selected as 1.0, making the proposed FOPI controller an integer PI controller. The proposed controller is compared with those of Padula et al. [27], Chen et al. [22], and Gude et al. [28]. Its ratio \( \omega_c/\omega_c^g = 3.39/1.70 \) and parameter \( \gamma = 1.40 \) were selected for \( M_p = 1.314 \).

Fig. 12 shows the output responses of the various tuning methods for disturbance rejection and set-point tracking. The proposed method's output responses showed the fastest setting time and the smallest overshoot. The method of Chen et al. [22] performed next best, while those of Padula et al. [27] and Gude et al. [28] provided slow responses with long settling times.

The proposed methods' controller setting parameters are listed

<table>
<thead>
<tr>
<th>Tuning methods</th>
<th>( K_C )</th>
<th>( \eta )</th>
<th>( \lambda )</th>
<th>( M_p )</th>
<th>( M_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed</td>
<td>0.386</td>
<td>13.156</td>
<td>1.1</td>
<td>1.047</td>
<td>1.72</td>
</tr>
<tr>
<td>F-MIGO</td>
<td>0.330</td>
<td>10.313</td>
<td>1.1</td>
<td>1.095</td>
<td>1.72</td>
</tr>
<tr>
<td>Lee et al. a</td>
<td>0.140</td>
<td>5.330</td>
<td>1.0</td>
<td>1.00</td>
<td>1.40</td>
</tr>
<tr>
<td>AMIGO</td>
<td>0.170</td>
<td>6.539</td>
<td>1.0</td>
<td>1.00</td>
<td>1.40</td>
</tr>
</tbody>
</table>

\( a = K_c(1 + (1/\eta))(1/as + 1), \) \( a = 0.1356, \) \( c_c = 22 \).
in Table 4, demonstrating its superiority. As can be seen from Fig. 13 and Table 4 that the TV values of the proposed method are bigger than those of the other but it still smooth enough for the properly operation.

The comparison between the FOPDT and the original fractional-order plant model as the other comparative methods. Although the detail results are omitted due to limited space, it should be noted that the proposed controller still provided good performances when the controller parameters were adjusted to have the same Ms values selected, as seen in the tables. However, although the detail results are not included here due to limited space, it should be noted that the proposed method still provided good performances when the controller parameters were adjusted to have the same Ms values as the other comparative methods.

5. Discussion

The proposed methodology provides the ratio of initial and crossover frequencies, \( \omega_i/\omega_c \), and the order of the ideal fractional feedback control system, \( \gamma \), as important user-defined tuning parameters. They are used here to control the tradeoff between performance and robustness. Therefore, it is necessary to provide guidelines that afford the best performance at a given degree of robustness for the different FOPI controllers.

Consider the previously reported fractional-order plant model for a heating furnace [29,30]:

\[
\hat{G}(s) = \frac{1}{14994s^{0.11} + 6009.5s^{0.87} + 1.69} \tag{37}
\]

To apply the proposed FOPI tuning rules, the above fractional system has to be approximated to the FOPDT by Oustaloup’s recursive approximation [6,14,31] which is widely used in fractional calculus. A generalized Oustaloup’s filter is designed by \( G_f(s) = K \prod_{k=1}^{N} \left( s + \alpha_k/\omega_k \right) \), where \( K, \alpha_k \), and \( \omega_k \) represent the gain, poles, and zeros, respectively which are calculated by \( K = \alpha_0 \sqrt{\omega_0/\omega_n} \), \( \alpha_k = \omega_0 \omega_k^{(2k+1)/N} \), and \( \omega_k = \sqrt{\omega_0/\omega_n} \). In this work, the order of the approximation is chosen as \( N=8 \), while \( \omega_0 \) and \( \omega_n \) are selected as 0.001 and 1000 rad/s, respectively. Eq. (38) is obtained by fitting the above-mentioned Oustaloup’s filter into a fixed model structure as the FOPDT.

\[
\hat{G}(s) = \frac{1.110e^{-32.1s}}{953.289s + 1} \tag{38}
\]

The comparison between the FOPDT and the original fractional-order model (Fig. 14) shows the effectiveness of the approximation.

a. Effect of Mp values on the tuning parameters and the closed-loop performance

To evaluate the effects of the tuning parameters by considering (38), the proposed controller settings are calculated for eleven
robustness levels, $M_p$, (Table 5), which demonstrate the following features of the proposed method:

- values of $\omega/\omega_{cg}$ and $\gamma$ can be easily obtained for any given $M_p$ based on (30). Note that the control system exhibits increased robustness at lower values of $M_p$; it exhibits better performance with less robust stability at higher values.

- Fig. 15 shows the effects of $M_p$ on the overall closed-loop performance as characterized by the IAE indices. The best closed-loop performance (smallest IAE value) can be achieved within the desirable range of $1.10 \leq M_p \leq 1.40$. Otherwise, poor closed-loop performances result. Therefore, it is necessary to determine the desirable values of $M_p$, $\omega/\omega_{cg}$, and $\gamma$ to set up a suitable tradeoff between performance and robustness in any given dynamic model.

b. Fractional order ($\lambda$) guideline for the proposed FOPI parameter tuning

In this work, the integral order ($\lambda$) was selected based on the guideline as shown in (21). In order to ensure that the guideline given in (21) is still valid for the proposed method, it is important to check its validation in a systematic way. Here, the proposed method is applied to three main kinds of dynamics for the FOPDT, which includes lag-dominated, delay-dominated, and balanced delay-lag dynamics, and the fractional order is scanned in the range of $0.6 \leq \lambda \leq 1.6$, $0.9 \leq \lambda \leq 1.2$, and $0.7 \leq \lambda \leq 1.3$, respectively. The best controller is then picked based on the IAE criterion with a given desired $M_p$ value.

Considering a FOPDT model given by (38) that represents a lag-dominated process with $\Delta = 0.0326$, the FOPI parameters are calculated using the proposed method for different $\lambda$ values with a desirable value of $M_p$ as 1.1. The resulting controller parameters and performance indices are listed in Table 5. The IAE values for set-point tracking.

### Table 4
Controller parameters and performance matrix for a lag and delay balanced process (Example 4).

<table>
<thead>
<tr>
<th>Tuning methods</th>
<th>$K_C$</th>
<th>$\tau$</th>
<th>$\lambda$</th>
<th>$M_p$</th>
<th>$M_s$</th>
<th>Set-point</th>
<th>Disturbance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed</td>
<td>1.18</td>
<td>1.14</td>
<td>1.0</td>
<td>1.314</td>
<td>2.112</td>
<td>1.404</td>
<td>0.213</td>
</tr>
<tr>
<td>Padula et al.</td>
<td>0.61</td>
<td>1.10</td>
<td>1.0</td>
<td>1.000</td>
<td>1.415</td>
<td>1.802</td>
<td>0.000</td>
</tr>
<tr>
<td>Chen et al.</td>
<td>0.74</td>
<td>0.71</td>
<td>1.0</td>
<td>1.315</td>
<td>1.888</td>
<td>1.739</td>
<td>0.232</td>
</tr>
<tr>
<td>Gude et al.</td>
<td>0.47</td>
<td>1.19</td>
<td>1.12</td>
<td>1.025</td>
<td>1.260</td>
<td>3.180</td>
<td>0.075</td>
</tr>
</tbody>
</table>

**IAE** denotes the IAE values for set-point tracking.

### Table 5
Controller parameters and performance matrix for various values of $M_p$.

<table>
<thead>
<tr>
<th>$M_p$</th>
<th>$\omega/\omega_{cg}$</th>
<th>$\gamma$</th>
<th>$K_C$</th>
<th>$\tau$</th>
<th>$\lambda$</th>
<th>IAE$_{sp}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.1010/0.0150</td>
<td>1.1600</td>
<td>9.690</td>
<td>103.085</td>
<td>0.70</td>
<td>88.86</td>
</tr>
<tr>
<td>1.1</td>
<td>0.1140/0.0215</td>
<td>1.1812</td>
<td>15.060</td>
<td>125.358</td>
<td>0.70</td>
<td>76.92</td>
</tr>
<tr>
<td>1.2</td>
<td>0.1294/0.0215</td>
<td>1.1383</td>
<td>16.400</td>
<td>131.300</td>
<td>0.70</td>
<td>77.13</td>
</tr>
<tr>
<td>1.3</td>
<td>0.1294/0.0225</td>
<td>1.1480</td>
<td>17.009</td>
<td>98.198</td>
<td>0.70</td>
<td>75.93</td>
</tr>
<tr>
<td>1.4</td>
<td>0.1300/0.0243</td>
<td>1.1550</td>
<td>18.640</td>
<td>103.556</td>
<td>0.70</td>
<td>78.45</td>
</tr>
<tr>
<td>1.5</td>
<td>0.1345/0.0244</td>
<td>1.1412</td>
<td>19.150</td>
<td>103.164</td>
<td>0.70</td>
<td>79.71</td>
</tr>
<tr>
<td>1.6</td>
<td>0.1340/0.0240</td>
<td>1.1470</td>
<td>19.569</td>
<td>95.459</td>
<td>0.70</td>
<td>79.91</td>
</tr>
<tr>
<td>1.7</td>
<td>0.1310/0.0267</td>
<td>1.1650</td>
<td>20.780</td>
<td>94.027</td>
<td>0.70</td>
<td>83.04</td>
</tr>
<tr>
<td>1.8</td>
<td>0.1310/0.0270</td>
<td>1.1683</td>
<td>21.022</td>
<td>90.290</td>
<td>0.70</td>
<td>84.28</td>
</tr>
<tr>
<td>1.9</td>
<td>0.1300/0.0280</td>
<td>1.1780</td>
<td>21.786</td>
<td>86.363</td>
<td>0.70</td>
<td>87.62</td>
</tr>
</tbody>
</table>

IAE$_{sp}$ denotes the IAE values for set-point tracking.

Fig. 13. Controller output responses of various controllers for lag and delay balanced process (example 4).

Fig. 14. Step responses of a fractional-order model and the FOPDT model for a heating furnace.

Fig. 15. Effects of $M_p$ on IAE for a heating furnace.
as shown in Figs. 3 and 11 respectively also indicate the validity of the lag-delay process (example 4 with a desirable value of $Mp$ as 1.31) and the balanced guideline given in [21] in the case of the lag-dominated controllers and performance matrix for various values of $\lambda$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$aw_{awd}$</th>
<th>$\gamma$</th>
<th>$K_C$</th>
<th>$\eta$</th>
<th>$Mp$</th>
<th>IAEmp</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.1142/0.02010</td>
<td>1.17800</td>
<td>13.426</td>
<td>41.341</td>
<td>1.1</td>
<td>97.00</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1140/0.02120</td>
<td>1.81820</td>
<td>14.632</td>
<td>78.843</td>
<td>1.1</td>
<td>90.33</td>
</tr>
<tr>
<td>0.7</td>
<td>0.1140/0.02150</td>
<td>1.81820</td>
<td>15.060</td>
<td>125.358</td>
<td>1.1</td>
<td>76.92</td>
</tr>
<tr>
<td>0.8</td>
<td>0.1135/0.02050</td>
<td>1.74900</td>
<td>14.372</td>
<td>170.268</td>
<td>1.1</td>
<td>81.02</td>
</tr>
<tr>
<td>0.9</td>
<td>0.1133/0.02097</td>
<td>1.17471</td>
<td>14.852</td>
<td>310.263</td>
<td>1.1</td>
<td>84.83</td>
</tr>
<tr>
<td>1.0</td>
<td>0.1115/0.02149</td>
<td>1.81200</td>
<td>15.227</td>
<td>483.014</td>
<td>1.1</td>
<td>92.61</td>
</tr>
<tr>
<td>1.1</td>
<td>0.1115/0.02527</td>
<td>1.21018</td>
<td>18.189</td>
<td>595.701</td>
<td>1.1</td>
<td>96.86</td>
</tr>
</tbody>
</table>

Fig. 16. Effects of fractional order ($\lambda$) on IAE for a heating furnace.

the guideline given in (21) in the case of the lag-dominated process. Our simulation studies for the delay-dominated process (example 1 with a desirable value of $Mp$ as 1.1) and the balanced lag-delay process (example 4 with a desirable value of $Mp$ as 1.31) as shown in Figs. 3 and 11 respectively also indicate the validity of the guideline given in (21) for the proposed method.

6. Conclusions

A systematic approach for the design of FOPI controllers is proposed. It is based on Bode's ideal transfer function and fractional calculus to give FOPI controllers analytically derived to provide improved performance for both disturbance rejection and set-point tracking.

Simulation examples demonstrate that the proposed method can be applied to a large number of dynamic models, consistently affording the enhanced performance with fast and well-balanced closed-loop time responses.

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References


